MATH327: StatMech and Thermo, Spring 2026

First homework assignment

Instructions

Complete all four questions below and submit your solutions by file upload on Canvas. Clear and neat presentations of your workings and the logic behind them will contribute to your mark. Use of resources beyond the module materials must be explicitly referenced in your submissions. This assignment is **due by 17:00 on Friday, 6 March 2026**. Anonymous marking is turned on, model solutions will be posted on Friday, 20 March 2026, and I will aim to return marks and feedback by Wednesday, 25 March 2026.

You should already be familiar with the Department's academic integrity guidance for 2025–2026, which states that by submitting solutions to this assessment you affirm that you have read and understood the Academic Integrity Policy detailed in Appendix L of the Code of Practice on Assessment, and that you have successfully passed the Academic Integrity Tutorial and Quiz in the course of your studies. You also affirm that the work you are submitting is your own and you have not copied material from another source nor committed plagiarism nor commissioned all or part of the work (including unacceptable proof-reading) nor fabricated, falsified or embellished data when completing your work. You also affirm that you have not colluded with any other student in the preparation and production of your work. You also affirm that any use of generative artificial intelligence (AI) software in relation to the assessment is in accordance with these instructions and the University Guidance. You also affirm that you have acted honestly, ethically and professionally in the preparation and production of your work. Marks achieved on this assessment remain provisional until they are ratified by the Board of Examiners in June 2026.

Question 1: Drift and diffusion

In February 2024 there was an oil spill near the Caribbean island of Tobago. Several million litres of oil moved from the site of the spill towards Grenada's territorial waters, $150\ \mathrm{km}$ away. We can analyze the motion of the oil by treating each droplet as a random walker moving in one dimension — towards or away from Grenadan waters. Satellite images and ocean models were used to estimate the rate at which the oil was moving. Suppose they indicated a drift velocity $v_{\mathrm{dr}} = 5\ \mathrm{km/hour}$ towards Grenadan waters, with a diffusion constant $D = 10\ \mathrm{km}/\sqrt{\mathrm{hour}}$.

How many hours did Grenada have in which to take action before 1% of the spilled oil was inside its waters? How much additional time did it take for the amount of oil inside Grenadan waters to double to 2% of the total?

`[20 marks]`

Suppose the oil were moving towards the island of Grenada itself, rather than its territorial waters. It is significantly more complicated to analyze the situation of oil washing up on Grenada's shores, because each droplet's random walk would *stop* once it reached the shore and left the water. This is known as a *first-passage process*. Without attempting this more complicated calculation, decide whether it will take more time, less time or the same amount time for the spilled oil to wash up on shore, compared to entering territorial waters, with everything else the same. Explain your choice with clear reasoning.

`[6 marks]`

Hint: The error function

$$\mathrm{erf}(u) = \frac{1}{\sqrt{\pi}} \int_{-u}^{u} e^{-x^2}\, dx \equiv P$$

may appear in your work, with $u > 0$. SciPy is one tool you can use to invert the error function to find $u = \mathrm{erf}^{-1}(P)$ for a given $0 < P < 1$. Here are some examples:

```
>>> import math
>>> from scipy import special
>>>
>>> sigmas = [0.682689492, 0.954499736, 0.997300204, \
...                         0.999936656, 0.999999427]
>>> for P in sigmas:
...   u = special.erfinv(P)
...   n = round(u * math.sqrt(2.0))
...   print("u = %.4f for P=%.7f (%d sigma)" % (u, P, n))

u = 0.7071 for P=0.6826895 (1 sigma)
u = 1.4142 for P=0.9544997 (2 sigma)
u = 2.1213 for P=0.9973002 (3 sigma)
u = 2.8284 for P=0.9999367 (4 sigma)
u = 3.5356 for P=0.9999994 (5 sigma)
```

Question 2: Heat capacity

(a) Starting from the expectation value for the internal energy in the canonical ensemble,

$$\langle E \rangle = \frac{1}{Z} \sum_{i=1}^{M} E_i \, e^{-\beta E_i},$$

derive a relation between the heat capacity

$$c_v = \frac{\partial}{\partial T} \langle E \rangle$$

and the quantity $\left\langle (E - \langle E \rangle)^2 \right\rangle$.

[10 marks]

(b) If the heat capacity vanishes at a non-zero temperature, what can we conclude about the micro-state energies E_i?

[4 marks]

(c) Derive a relation between c_v, $\frac{\partial}{\partial T} c_v$ and the quantity $\left\langle (E - \langle E \rangle)^3 \right\rangle$.

[10 marks]

Question 3: Ideal gas

Consider a classical ideal gas of N indistinguishable particles at fixed temperature T. Each particle has fixed mass m and energy

$$E(\vec{p}, \vec{r}) = \frac{1}{2m} p^2 + \frac{\alpha}{2} r^2,$$

where p^2 and r^2 are the inner products of its momentum $\vec{p} = (p_x, p_y, p_z)$ and position $\vec{r} = (x, y, z)$, respectively, and α is a real positive constant. The single-particle partition function is

$$Z_1(T) = \int \exp\left[-E(\vec{p}, \vec{r})/T\right] \, d^3p \, d^3x.$$

(a) What is the Helmholtz free energy for the N indistinguishable particles?

[3 marks]

(b) Calculate the internal energy $\langle E \rangle$ and the entropy S.

[5 marks]

(c) Calculate $\langle \vec{r} \rangle$ and $\langle r^2 \rangle$.

[8 marks]

(d) Calculate $\langle \vec{p} \rangle$ and $\langle p^2 \rangle$.

[8 marks]

Question 4: Energy and entropy

Using the canonical ensemble with inverse temperature $\beta = 1/T$, consider a system of $N \gg 1$ indistinguishable non-interacting spins in a background magnetic field with strength $H > 0$.

(a) What are the internal energy $\langle E \rangle_I$ and entropy S_I as functions of β, N and H?

[4 marks]

(b) Use your results to confirm the following low- and high-temperature limits:

$$\lim_{T\to 0} \langle E \rangle_I = -NH \qquad \lim_{T\to 0} S_I = 0$$
$$\lim_{T\to \infty} \langle E \rangle_I = 0 \qquad \lim_{T\to \infty} S_I = \log(N+1)$$

[6 marks]

(c) For low but non-zero temperatures, expand both $\langle E \rangle_I$ and S_I in terms of $\varepsilon \equiv e^{-2\beta H} \ll 1$. Show that the largest temperature-dependent term in the energy expansion is proportional to ε and find the constant of proportionality. Show that the largest temperature-dependent term in the entropy expansion is proportional to $\beta\varepsilon$ and find the constant of proportionality.

[8 marks]

(d) For high but finite temperatures, expand both $\langle E \rangle_I$ and S_I in terms of $x \equiv 2\beta H \ll 1$. Show that the largest temperature-dependent term in the energy expansion is proportional to x and find the constant of proportionality. Show that the largest temperature-dependent term in the entropy expansion is proportional to x^2 and find the constant of proportionality.

[8 marks]

The following famous results may be useful:

$$\frac{1}{1-e^{-x}} = \frac{1}{x} + \frac{1}{2} + \frac{x}{12} - \frac{x^3}{720} + \frac{x^5}{30\,240} + \mathcal{O}(x^6)$$
$$\log\left[1-e^{-x}\right] = \log(x) - \frac{x}{2} + \frac{x^2}{24} - \frac{x^4}{2880} + \frac{x^6}{181\,440} + \mathcal{O}(x^7)\,.$$