MATH327: StatMech and Thermo

Tuesday, 5 May 2026

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Something to consider

The mean-field approach

provides a non-interacting approximation to the Ising model

How reliable is this approximation? What determines its reliability?

Recap

Ising model magnetization as order parameter

transition vs. crossover

Mean-field approx. $\to$ self-consistency condition

$\langle m \rangle = \tanh(\beta(2d\langle m \rangle + H))$

Plan

Mean-field $H=0$ transition

critical exponent

reliability

Recall mean-field solution ($d=2$ $T=4$)

$H=0 \to \langle m \rangle = 0$ (disordered)

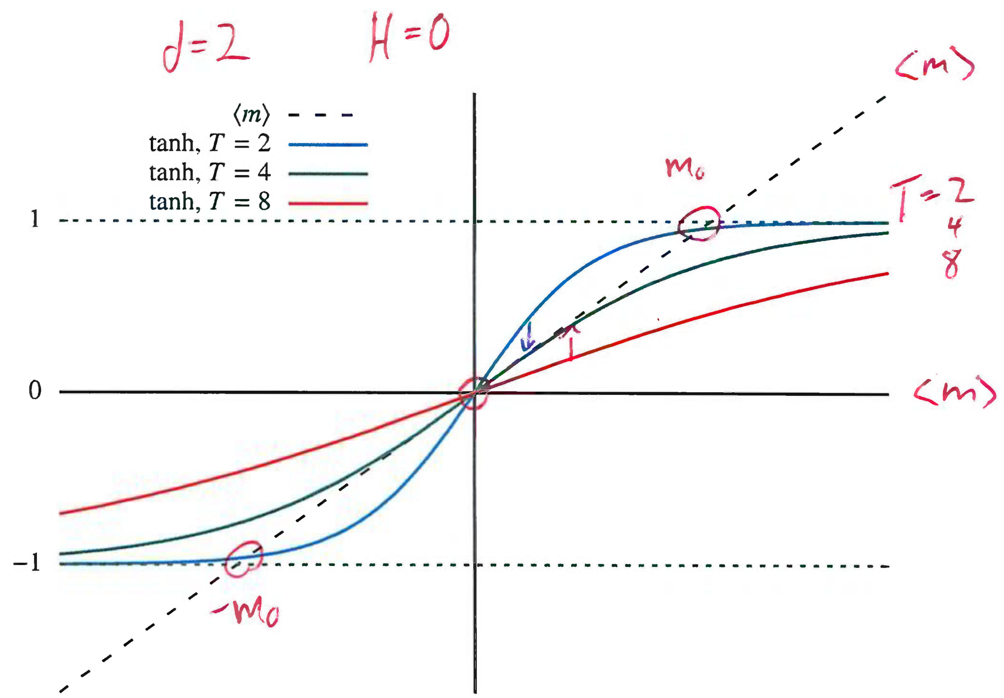
$H \neq 0 \to \langle m \rangle = \pm m_0 \neq 0$ (ordered)

$\hookrightarrow m_0 \to 1$ as $T \to 0$

Expect low-T ordered phase even with $H=0$

Compare $T = 8, 4, 2$

$\langle m \rangle = 0$ always possible



d=2
H=0
⟨m⟩
tanh, T = 2
tanh, T = 4
tanh, T = 8
1
0
−1
m₀
−m₀
T=2
4
8
⟨m⟩
⟨m⟩

Lower T → additional $\langle m \rangle = \pm m_0 \neq 0$
Which is true solution realized by the system?

Consider perturbing $\langle m \rangle = 0 \rightarrow \langle m \rangle = \varepsilon > 0$

$T = 8$: $\langle m \rangle$ too large compared to tanh
→ decrease to stable $\langle m \rangle = 0$

$T = 2$: $\langle m \rangle$ too small
→ increase to stable $\langle m \rangle = m_0 > 0$
Similarly $-\varepsilon \rightarrow -m_0$
Conclude $\langle m \rangle = 0$ unstable
→ ordered phase for low T ✓

Alternative visualization
Plot $\tanh(2\beta d \langle m \rangle) - \langle m \rangle$, find zeros
Positive → $\langle m \rangle$ too small
Negative → $\langle m \rangle$ too large

∴ $H = 0$ mean-field approx. gives expected phases
When does $\langle m \rangle = 0$ become unstable?
Need tanh > $\langle m \rangle$ for $\langle m \rangle = \varepsilon > 0$
→ slope > 1 at $\langle m \rangle = 0$

$$\boxed{\begin{aligned} \frac{d}{d\langle m \rangle} \tanh(2\beta d \langle m \rangle)\Big|_{\langle m \rangle = 0} &= \frac{d}{d\langle m \rangle}\left[2\beta d \langle m \rangle + \mathcal{O}(\langle m \rangle^3)\right]_{\langle m \rangle = 0} \\ &= 2\beta d = 1 \rightarrow T_c = \frac{1}{\beta_c} = 2d \end{aligned}}$$

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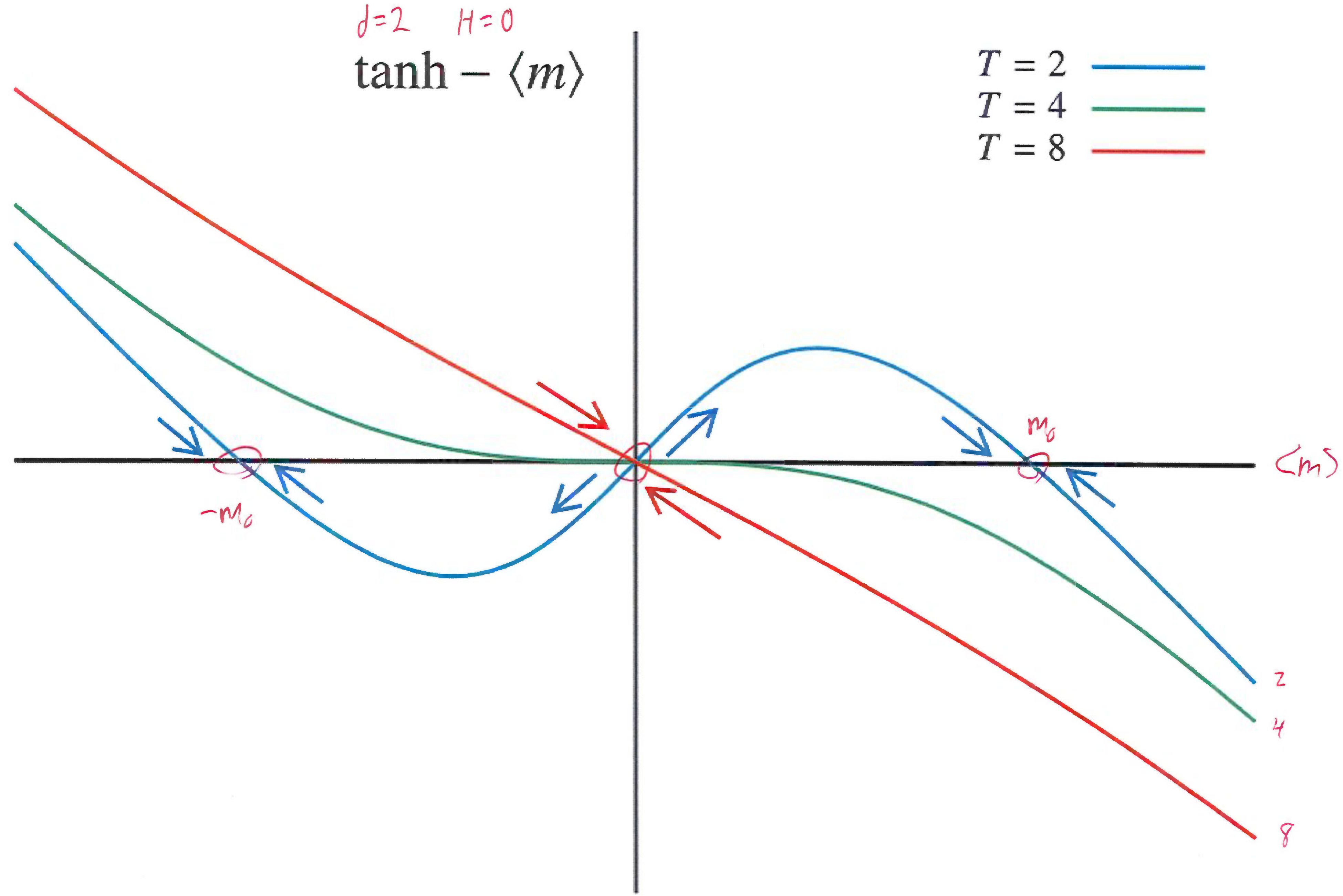
Is this a phase trans. with discontinuity?



d=2 H=0
tanh − ⟨m⟩
T = 2
T = 4
T = 8
−m₀
m₀
⟨m⟩
2
4
8

Consider $T \lesssim T_c$ so that $0 < |\langle m \rangle| \ll 1$

$$\langle \cancel{m} \rangle = \tanh(2\beta d \langle m \rangle) = 2\beta d \langle \cancel{m} \rangle - \frac{1}{3}(2\beta d \langle m \rangle)^3 + O(\langle m \rangle^5)$$

$$\frac{1}{3}\left(\frac{T_c}{T}\right)^3 \langle m \rangle^2 = \frac{T_c}{T} - 1$$

$$\langle m \rangle^2 = 3\left(\frac{T}{T_c}\right)^3 \left(\frac{T_c}{T} - 1\right) = 3\left(\cancel{\frac{T}{T_c}}\right)^2 \left(1 - \frac{T}{T_c}\right)$$

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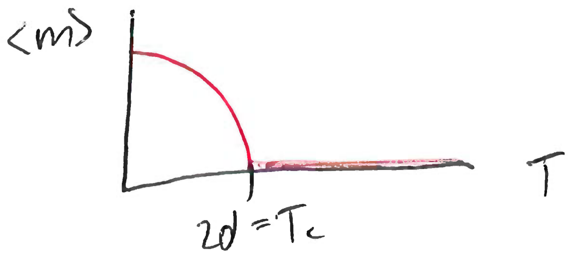
$$\frac{T}{T_c} \approx 1 \rightarrow \langle m \rangle \approx \pm\sqrt{3}\left(1 - \frac{T}{T_c}\right)^{1/2} = \pm\sqrt{3}\left(\frac{T_c - T}{T}\right)^{1/2}$$

So OP continuous:

$$\langle m \rangle \propto \begin{cases} (T_c - T)^{1/2} & \text{for } T \lesssim T_c \\ 0 & \text{for } T \gtrsim T_c \end{cases}$$



But $\frac{d\langle m \rangle}{T} \propto \frac{1}{(T_c - T)^{1/2}}$ for $T \lesssim T_c$

divergence as $T \to T_c$ from below

$\therefore$ Mean-field approx predicts second-order PT for Ising model

Occur whenever $OP \propto (T_c - T)^b$

with <u>critical exponent</u> $0 < b < 1$

Many phase trans. have same sets of critical exponents

<u>Universality</u>: Emergent behaviour near critical point independent of microscopic details

Is mean-field prediction correct? (2nd order, $T_c = 2d$, $b = 1/2$)

$d=1$: No phase trans. (Ising 1924) ✗

$d=2$: 2nd order PT (Onsager 1944) ✓

$$T_c = \frac{2}{\log(1+\sqrt{2})} \simeq 2.27$$

So MF $T_c = 4$ off by $\sim 2\times$

$b = \frac{1}{8}$, " $b = 1/2$ off by $4\times$

$d=3$: Numerical analyses → 2nd-order PT

$T_c \simeq 4.5$ (vs. 6)

$b \simeq 0.32$ (vs. 0.5)

$d \geq 4$: 2nd-order PT with $b = \frac{1}{2}$ ✓

$T_c \to 2d$ as $d \to \infty$

Mean-field ~~approx~~ becomes exact

(more n.n. → more reliable)