Mon 3 Feb

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Recap

Probability spaces

Law of large numbers $\mu \approx \overline{X}_R$ $R \gg 1$

Central limit theorem $p(x)$ gaussian

single-expt, μ & σ^2

Today

Random walk

Law of diffusion

Simple random walk

Step only right $(+\ell)$ prob, p or left $(-\ell)$ prob $q=1-p$ along line

Set $\ell=1$ and step every δt (time interval)

N step $\rightarrow$ total time $t = N\delta t$

LRLRRR $N=6$

start $x_0=0$, final $x=2=\sum_{i=1}^{N} x_i$

RLRLLL $\rightarrow x=-2$

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# of walks $2 \cdot 2 \cdots 2 = 2^N \rightarrow 64$

$P(LRLRRR) = qpqppp = p^4 q^2$

General N-step walk w/ r steps to right, $N-r$ to left

$\rightarrow$ prob $\sim p^r q^{N-r}$

Order doesn't matter

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$N=6$, $x=4$ RRRRRL, RRRRLR, ...

$6 = \binom{6}{5} = \binom{N}{r} \rightarrow$ overall $P_r = \binom{N}{r} p^r q^{N-r}$

Compute expected final position $\langle x \rangle = \sum_x x\, P(x)$

Fluctuations $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

Convert $x, P(x) \longleftrightarrow r, P_r$ for special case $l=1$

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$$\boxed{x = (+1)r + (-1)(N-r) = 2r - N}$$

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$$\langle x \rangle = \sum_{r=0}^{N} (2r-N)\, P_r = 2\langle r \rangle - N$$

$$\langle x^2 \rangle = \sum_r (2r-N)^2 P_r = 4\langle r^2 \rangle - 4N\langle r \rangle + N^2$$

Trick to get $\langle r \rangle, \langle r^2 \rangle, \ldots$: generating function

$$G(\theta) = \sum_{r=0}^{N} e^{r\theta} P_r \qquad G(0) = 1$$

$$\frac{d}{d\theta} G(\theta)\Big|_{\theta=0} = \sum_r \frac{d}{d\theta} e^{r\theta} P_r\Big|_{\theta=0} = \sum_r r \underbrace{e^{r\theta}}_{1} P_r\Big|_{\theta=0} = \langle r \rangle$$

$$\frac{d^n}{d\theta^n} G(\theta)\Big|_{\theta=0} = \sum_r r^n \underbrace{e^{r\theta}}_{1} P_r\Big|_{\theta=0} = \langle r^n \rangle$$

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Plug in P_r: $G(\theta) = \sum_r e^{r\theta} \binom{N}{r} p^r q^{N-r}$

$$= \sum_r \binom{N}{r} (e^\theta p)^r q^{N-r}$$

Binomial Formula: $(a+b)^N = \sum_i \binom{N}{i} a^i b^{N-i}$

$$\therefore G(\theta) = (e^\theta p + q)^N$$

$$G(0) = (p+q)^N = 1 \checkmark$$

$$\text{So } \langle r\rangle = \frac{d}{d\theta}(e^\theta p+q)^N\Big|_{\theta=0} = N(e^\theta p+q)^{N-1}e^\theta p\Big|_{\theta=0}$$
$$= Np \quad ✓$$

$$\langle r^2\rangle = \frac{d}{d\theta}\left(N e^\theta p\,(e^\theta p+q)^{N-1}\right)_{\theta=0}$$

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$$= \left[Ne^\theta p\,(e^\theta p+q)^{N-1} + N(N-1)e^\theta p(e^\theta p+q)^{N-2}e^\theta p\right]_{\theta=0}$$
$$= Np + N(N-1)p^2 = Np(1+Np-p)$$
$$= Np(Np+q)$$

Final result:

$$\langle x\rangle = 2\langle r\rangle - N = 2Np - N = N(2p-1)$$

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$$
$$= 4\langle r^2\rangle - \cancel{4N\langle r\rangle} + \cancel{N^2} - (2\langle r\rangle - N)^2$$
$$\hookrightarrow 4\langle r\rangle^2 - \cancel{4N\langle r\rangle} + \cancel{N^2}$$

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$$\Delta x = \sqrt{4(\langle r^2\rangle - \langle r\rangle^2)}$$
$$= 2\sqrt{\cancel{N^2p^2} + Npq - \cancel{N^2p^2}} = 2\sqrt{Npq}$$

Sanity checks

p	$\langle x\rangle$	Δx
0	$-N$	0
1	N	0
1/2	0	$\sqrt{N}$

✓

$\Delta x \propto \sqrt{N}$ generic ...

Diffusion

Recall $t = N\,\delta t \quad \longleftrightarrow \quad N = \frac{t}{\delta t}$

$$\langle x \rangle = \frac{t}{\delta t}(2p-1) = \frac{2p-1}{\delta t}\, t = V_{dr}\, t$$

$V_{dr} = \frac{\langle x \rangle}{t}$ is <u>drift velocity</u> of expected final position

Fluctuations $\underbrace{\Delta x}_{\text{diffusion length}} = 2\sqrt{\frac{pq}{\delta t}}\,\sqrt{t} = \underbrace{D}_{\text{diffusion constant}}\sqrt{t}$ is <u>law of diffusion</u>