MATH327: Statistical Physics, Spring 2024

Tutorial activity — Probabilities

This activity is for our 1 February tutorial, but at that point we won't yet have had enough lecture time to introduce all the concepts it involves. We'll begin the tutorial by completing our definition of probability spaces and introducing the roulette wheel (pages 10–13 of the lecture notes). At that point you can start working on the first two parts below. In the next lecture or two we'll cover the central limit theorem, and you can move on to the final two parts. We'll review the activity during one of our upcoming computer labs, which replace tutorials for the next three weeks.

Consider a simple game of roulette in which we just place bets on whether or not the ball will end up in a red- or black-coloured pocket. If we bet correctly we get back twice the money we put in; otherwise we lose our money. We define our (potentially negative) *gain* to be the amount we receive minus the amount we spend on bets.

- Suppose we place £5 bets on 'black' for each of N spins of the roulette wheel. What are the probabilities and gains of winning and of losing for any single one of those spins? Letting $W = 0, \cdots, N$ be the number of spins where we win, what is our total gain G_W as a function of (N, W)?

- Recall that the number of different ways we could win W times out of N attempts is given by the binomial coefficient

$$\binom{N}{W} = \frac{N!}{W!\,(N-W)!},$$

with $0! = 1$. Setting $N = 5$, what are the six probabilities p_0 through p_5 that we win $W = 0, \cdots, 5$ times? What is the general p_W for any (N, W)?

- Now let's apply the central limit theorem to this setup. What are the mean gain and its variance for a single spin of the wheel? What is the resulting probability distribution $p(g)$ given by the central limit theorem for the gain $g \in \mathbb{R}$ after $N \gg 1$ spins?

- In order to compare the distribution $p(g)$ against the probabilities p_W considered above, we need to integrate $p(g)$ over appropriate intervals as discussed in class (and on page 16 of the lecture notes). Natural intervals to consider are

$$P_{\text{integ}}(G_W) \equiv \int_{G_W-\Delta G/2}^{G_W+\Delta G/2} p(g)\ dg,$$

  where $\Delta G = G_{W+1} - G_W$ is a constant you can read off from your work so far. These numerical integrations are not convenient to do by hand, but can easily be performed by Maple, Python, MATLAB, Mathematica, etc. Alternately, we can simplify further by approximating $p(g)$ as a constant within each interval, which would give us

$$P_{\text{const}}(G_W) \equiv p(G_W)\ \Delta G.$$

  Keeping $N = 5$, what are the six P_{integ} and P_{const}?