MATH327: Statistical Physics, Spring 2023

Homework assignment 2

Instructions

Complete all four questions below and submit your solutions by file upload on Canvas. By submitting solutions to this assessment you affirm that you have read and understood the Academic Integrity Policy detailed in Appendix L of the Code of Practice on Assessment and have successfully passed the Academic Integrity Tutorial and Quiz. The marks achieved on this assessment remain provisional until they are ratified by the Board of Examiners in June 2023. Clear and neat presentations of your workings and the logic behind them will contribute to your mark. This assignment is **due by 17:00 on Wednesday, 3 May**. Anonymous marking is turned on and I will aim to return feedback by 14 May.

Question 1: Indistinguishable spins

In Section 3.4.2 of the lecture notes, we computed the Helmholtz free energy for $N \gg 1$ indistinguishable spins (Eq. 45):

$$F_I(\beta) = -NH - \frac{\log\left[1 - e^{-2(N+1)\beta H}\right]}{\beta} + \frac{\log\left[1 - e^{-2\beta H}\right]}{\beta}.$$

What are the corresponding expressions for the internal energy $\langle E \rangle_I$ and the entropy S_I?

`[6 marks]`

What are the first **two** non-zero terms in **each** low-temperature ($e^{-2\beta H} \ll 1$) expansion of $\langle E \rangle_I$ and S_I?

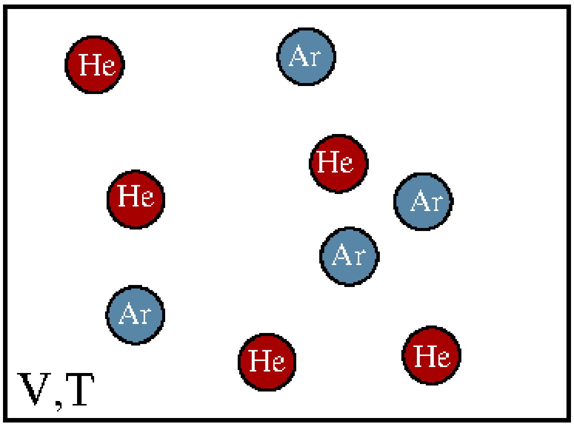
`[12 marks]`

What is the first non-zero term in the high-temperature ($\beta H \ll 1$) expansion of $\langle E \rangle_I$? What are the first **two** non-zero terms in the high-temperature expansion of S_I?

`[12 marks]`

Question 2: Mixed ideal gases

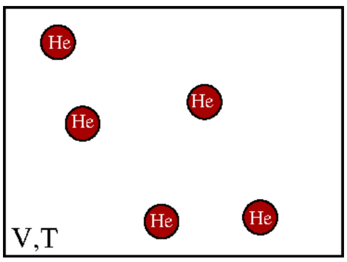
Consider a mixture of two classical ideal gases in thermodynamic equilibrium, in a container of volume V at temperature T, like that illustrated below. Let N_1 and N_2 be the fixed particle numbers of the two gases. Within each gas the particles are indistinguishable, but particles of one gas are distinguishable from particles of the other gas. In particular, they have different masses m_1 and m_2, implying different thermal de Broglie wavelengths λ_1 and λ_2.

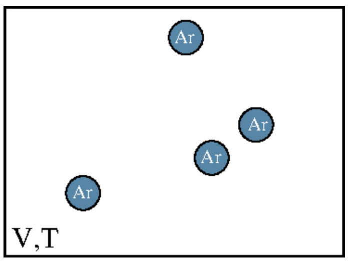


(a) Calculate the canonical partition function Z and the Helmholtz free energy of the $(N_1 + N_2)$-particle mixture, approximating $\log(N_i!) \approx N_i \log N_i - N_i$.
`[6 marks]`

(b) Calculate the internal energy $\langle E \rangle$ and the entropy S of the mixture. What is the condition of constant entropy?
`[6 marks]`

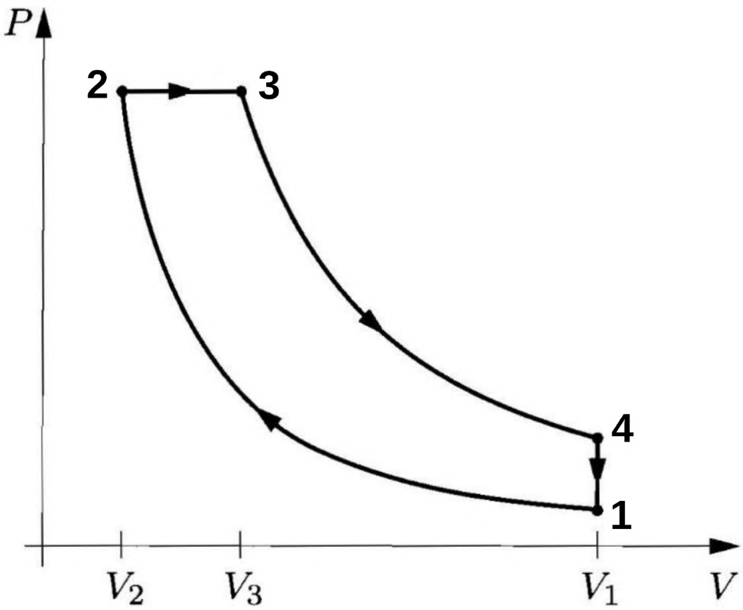
(c) Calculate the pressure P of the mixture, and relate it to the pressures P_1 and P_2 of each gas in isolation (as illustrated below).
`[6 marks]`





Question 3: Thermodynamic cycle

Consider the Diesel cycle defined by the PV diagram shown below, in which the 'compression' stage $1 \to 2$ and the 'power' stage $3 \to 4$ are both adiabatic, while the pressure is constant during the 'injection/ignition' stage $2 \to 3$, and the volume is constant during the 'exhaust' stage $4 \to 1$.



Calculate the efficiency of the Diesel cycle, η_D, in terms of the compression ratio $r \equiv V_1/V_2 > 1$ and the cutoff ratio $C \equiv V_3/V_2 > 1$, where $C < r$.

[20 marks]

Fixing the compression ratio r, compare η_D to the efficiency of the Otto cycle. Is the Diesel cycle more efficient than the Otto cycle, less efficient, or the same? How does this comparison depend on the cutoff ratio C?

[8 marks]

Question 4: Magnetization

Consider a classical system of N distinguishable, non-interacting 'spins' in a lattice at temperature $T = 1/\beta$, where the value s_n of each spin can vary *continuously* in the range $-1 \leq s_n \leq 1$. In an external magnetic field of strength $H > 0$, the internal energy of the system is $E = -H \sum_{n=1}^{N} s_n$.

(a) Calculate the canonical partition function Z and the Helmholtz free energy F of the system, both as functions of βH.

Hint: Just like the continuous momenta considered in Sections 4.1 and 8.1 of the lecture notes, you will need to integrate over the continuous s_n.

[8 marks]

(b) The derivative of the Helmholtz free energy with respect to the magnetic field defines the magnetization

$$\langle m \rangle = -\frac{1}{N} \frac{\partial F}{\partial H}.$$

Assuming finite $H > 0$, calculate $\langle m \rangle$ for this system as a function of βH, and determine its low- and high-temperature limits, $\lim_{T \to 0} \langle m \rangle$ and $\lim_{T \to \infty} \langle m \rangle$.

[8 marks]

(c) Calculate the leading T-dependent correction to **each** of the low- and high-temperature limits of $\langle m \rangle$ from the previous part.

[8 marks]