MATH327: Statistical Physics, Spring 2023

Tutorial activity — Stirling's formula

We have already made use of Stirling's formula in the following form:

$$\log(N!) = N \log N - N + \mathcal{O}(\log N) \approx N \log N - N \qquad \text{for } N \gg 1,$$

which implies

$$N! \approx \exp\left[N \log N - N\right] = \left(\frac{N}{e}\right)^N.$$

This can be made more precise:

$$N! = \sqrt{2\pi N}\left(\frac{N}{e}\right)^N \left(1 + \frac{A}{N} + \frac{B}{N^2} + \frac{C}{N^3} + \cdots\right) \tag{1}$$

with calculable coefficients A, B, C, etc.[1] By performing a sequence of analyses of increasing complexity, we can build up these results.

First analysis: Derive the bounds

$$N \log N - N < \log(N!) < N \log N \tag{2}$$

for $N \gg 1$. The second bound is the easier one. There are multiple ways to obtain the first bound. One pleasant approach is to consider the series expansion for e^x. Together, these bounds establish

$$1 - \frac{1}{\log N} < \frac{\log(N!)}{N \log N} < 1 \qquad \Longrightarrow \qquad \log(N!) \sim N \log N$$

Second analysis: Compute the first term in Eq. 1, $N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$. This requires several steps, the first of which is to consider the **gamma function**

$$\Gamma(N+1) \equiv \int_0^\infty x^N e^{-x}\, dx.$$

Show that $\Gamma(N+1) = N!$ for integer $N \geq 0$. In other words, derive the Euler integral (of the second kind)

$$N! = \int_0^\infty x^N e^{-x}\, dx. \tag{3}$$

Again, this can be done in multiple ways, including induction with integration by parts or by taking derivatives of

$$\int_0^\infty e^{-ax}\, dx = a^{-1}$$

and then setting $a = 1$.

[1] James Stirling computed the $\sqrt{2\pi}$ while Abraham de Moivre derived the expansion in powers of $1/N$. An interesting aspect of this expansion is that it is **asymptotic** — it has a vanishing radius of convergence but can provide precise approximations if truncated at an appropriate power.

The next step in this second analysis is to approximate the gamma function as a gaussian integral. Show that the integrand $x^N e^{-x} = \exp\left[N \log x - x\right]$ of Eq. 3 is maximized at $x = N$.

For $N \gg 1$, the integrand is sharply peaked around this maximum at $x = N$. You can check this for yourself or take it as given. We can therefore focus on a small region around this peak by changing variables to $y \equiv x - N$ and considering $\left|\frac{y}{N}\right| \ll 1$. Expand the $\log x$ in the integrand, up to and including terms quadratic in $\frac{y}{N}$. You should be left with the desired result, except for the following factor, which can be approximated by a gaussian integral (note the lower bound of integration):

$$\int_{-N}^{\infty} e^{-y^2/(2N)}\,dy \approx \int_{-\infty}^{\infty} e^{-y^2/(2N)} = \sqrt{2\pi N}.$$

The error introduced by extending the integration from $(-N, \infty)$ to $(-\infty, \infty)$ is exponentially small and could be captured by computing the series of corrections suppressed by powers of $\frac{1}{N}$ in Eq. 1.

This leads us to the **third analysis**: Compute some of the leading power-suppressed corrections in Eq. 1. That is, determine the coefficients A, B, etc. Again, there are many ways to achieve this, including higher-order expansions of the $\log x$ considered above. One pleasant approach is to compare $N!$ and $(N+1)!$, now that we have derived the series prefactor $\sqrt{2\pi N}\left(\frac{N}{e}\right)^N$.