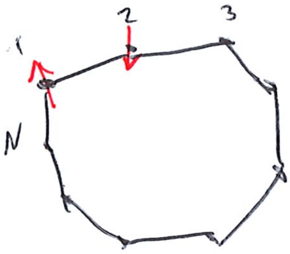
Supplement: Ising model exact results (12 May)

Can exactly solve 1d Ising model (1924)

$d \cdot N = N$ links $\quad \sum_{(jk)} s_j s_k = \sum_n s_n s_{n+1}$ $\quad$ PBC $s_{N+1} = s_1$



$$E = -\sum_n \left[s_n s_{n+1} + \frac{H}{2}(s_n + s_{n+1}) \right]$$

$$Z = \sum_{\{s_n\}} e^{-\beta E} = \left(\sum_{s_1=\pm 1} \cdots \sum_{s_N=\pm 1} \right) \prod_n \exp\left[\beta s_n s_{n+1} + \frac{\beta H}{2}(s_n + s_{n+1}) \right]$$

Want to factorize

but $\underline{\text{same}}$ s_j appears for $n=j$ and $n+1=j$

Trick: Organize as matrix multiplication with

$$\{s_n, s_{n+1}\} = \begin{pmatrix} \{1,1\} & \{1,-1\} \\ \{-1,1\} & \{-1,-1\} \end{pmatrix}$$

$$\exp \rightarrow T_n = \begin{pmatrix} e^{\beta} e^{\beta H} & e^{-\beta} \\ e^{-\beta} & e^{\beta} e^{-\beta H} \end{pmatrix}$$

Matrix product $T_n \cdot T_{n+1}$ diagonal holds all configs w/ fixed s_{n+1}

$$\longrightarrow Z = \mathrm{Tr}\left[\prod_n T_n \right] = \mathrm{Tr}\left[T^N \right] = \sum_{i,k,l,\ldots,N} T_{ij} T_{jk} T_{kl} \cdots T_{Ni}$$

$\underline{\text{transfer matrix}}$

For diagonal $T = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$ $\quad Z = \mathrm{Tr}\begin{pmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{pmatrix} = \lambda_+^N + \lambda_-^N$

Diagonalize: $\det(T - \lambda \mathbb{I}) = 0$

$$\begin{vmatrix} e^{\beta} e^{\beta H} - \lambda & e^{-\beta} \\ e^{-\beta} & e^{\beta} e^{-\beta H} - \lambda \end{vmatrix} = \lambda^2 - \lambda e^{\beta}\left(e^{\beta H} + e^{-\beta H}\right) + e^{2\beta} - e^{-2\beta}$$

$$= \lambda^2 - 2\lambda e^{\beta} \cosh(\beta H) + 2\sinh(2\beta) = 0$$

$$Ax^2 + Bx + C = 0$$

Eigenvalues: $$\lambda_\pm = \frac{\cancel{1}}{\cancel{2}}\left(\cancel{2}e^{\beta}\cosh(\beta H) \pm \sqrt{\cancel{4}e^{2\beta}\cosh^2(\beta H) - \cancel{8}_{2}\sinh(2\beta)}\right)$$

$$= e^{\beta}\cosh(\beta H)\left[1 \pm \sqrt{1 - \frac{2\sinh(2\beta)}{e^{2\beta}\cosh^2(\beta H)}}\right]$$

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Check: $\beta \geq 0$ and $H \geq 0 \longrightarrow$ real eigenvalues, $\lambda_+ > \lambda_-$

$\beta \to 0$ (high-T) $\quad \lambda_+ \to 1\left[1+\sqrt{1-\frac{0}{1}}\right] = 2$

$\lambda_- \to 0$

$$H=0 \qquad \lambda_+ = e^{\beta}\left[1+\sqrt{\cancel{1}-\left(\frac{\cancel{e^{2\beta}}-e^{-2\beta}}{e^{2\beta}}\right)}\right] = e^{\beta}\left(1+\sqrt{e^{-4\beta}}\right) = e^{\beta}+e^{-\beta}$$
$$= 2\cosh\beta$$

$$\lambda_- = e^{\beta} - e^{-\beta} = 2\sinh\beta$$

With $\frac{\lambda_-}{\lambda_+} < 1$ and $N \gg 1$

$$Z = \lambda_+^N + \lambda_-^N = \lambda_+^N\left(1+\left(\frac{\lambda_-}{\lambda_+}\right)^N\right) \to \underline{\lambda_+^N}$$

$$Z = e^{N\beta}\cosh^N(\beta H)\left[1+\sqrt{1-\frac{2\sinh(2\beta)}{e^{2\beta}\cosh^2(\beta H)}}\right]^N$$

Compute $H=0$ magnetization

$$\langle m\rangle = \frac{1}{N\beta}\frac{\partial}{\partial H}\log Z\bigg|_{H=0} = \frac{1}{\lambda_+\beta}\frac{\partial\lambda_+}{\partial H}\bigg|_{H=0}$$

$$\frac{\partial}{\partial H}\left[e^{\beta}\cosh(\beta H)+\sqrt{e^{2\beta}\cosh^2(\beta H)-2\sinh(2\beta)}\right]_{\underline{H=0}} \propto \sinh(\beta H) \to 0$$

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So $\langle m\rangle = 0$ for all temperatures

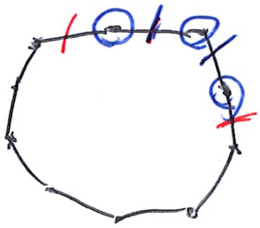
└ always disordered, no phase transition!

Why? Recall First excited state lower prob. × higher degeneracy

$d=1$ can flip single spin

n.n. pair of spins

3, 4, 5, ... spins in a row



Magnetization $-1+\frac{2}{N}, \ldots, 0, \ldots, 1-\frac{2}{N}$

$\rightarrow 0 \rightarrow$ disordered

Useful to analyze in terms of domain walls

separating up & down spins

Walls move freely without changing energy

while changing magnetization

Consider 2d with $H=0$

Kramers-Wannier duality (1941)

Remarkable relation between two 2d $H=0$ Ising model part. funcs

$$\frac{Z(\beta)}{2^N \cosh^{2N}\beta} = \frac{Z(\tilde{\beta})}{2e^{2N\tilde{\beta}}} \qquad \text{where } \sinh(2\beta) = \frac{1}{\sinh(2\tilde{\beta})}$$

high-T $\Leftrightarrow$ low-T

disordered ordered

Early example of duality like AdS/CFT

Suppose there is a phase transition at $T_c = 1/\beta_c$

where ordered & disordered phases coincide

Must have $\beta_c = \tilde{\beta}_c \longrightarrow \sinh^2(2\beta_c) = 1$

$$\beta_c = \frac{1}{2}\operatorname{arcsinh}(1) = \frac{1}{2}\log(1+\sqrt{2}) \simeq 0.44$$

$\operatorname{arcsinh}(x) = \log(x + \sqrt{x^2+1})$

Exact $T_c = \dfrac{2}{\log(1+\sqrt{2})} \simeq 2.27$ reproduced by Onsager (1944)