MATH327: Statistical Physics

Tuesday, 9 May 2023

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Something to consider

We have seen how the Debye model converts complicated interactions among atoms into a non-interacting gas of phonons.

How can we do something similar for the simpler interactions among spins in the Ising model?

Recap

Ising model $\quad E = -\sum_{(jk)} s_j s_k - H \sum_n s_n$

Magnetization $\langle m \rangle = \frac{1}{N\beta} \frac{\partial}{\partial H} \log Z$ is order parameter

distinguishing high-T disordered phase $\langle |m| \rangle \to 0$

vs. low-T ordered phase $\langle |m| \rangle \to 1$

Discontinuity $(N \to \infty) \longleftrightarrow$ phase transition

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Consider $\langle m \rangle = \frac{1}{N} \sum_n \langle s_n \rangle$ is average (mean) value of spin (config. independent)

Expand interactions

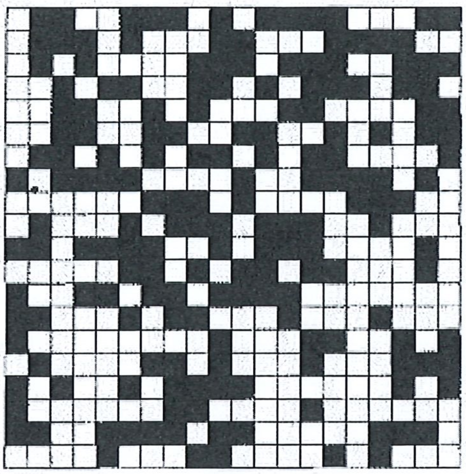
$$s_j s_k = \left[(s_j - \langle m \rangle) + \langle m \rangle \right] \times \left[(s_k - \langle m \rangle) + \langle m \rangle \right]$$

$$= (s_j - \langle m \rangle)(s_k - \langle m \rangle) + (s_j + s_k)\langle m \rangle + \langle m \rangle^2$$

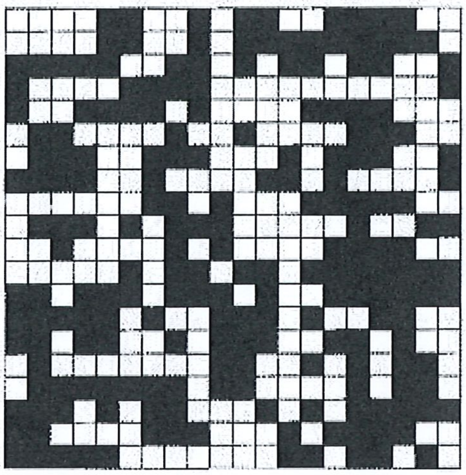
negligible

Suppose on average only small fluctuations around mean spin

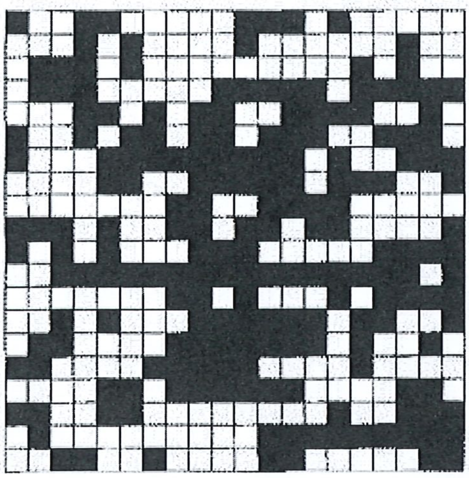
$\langle M \rangle \sim 0$



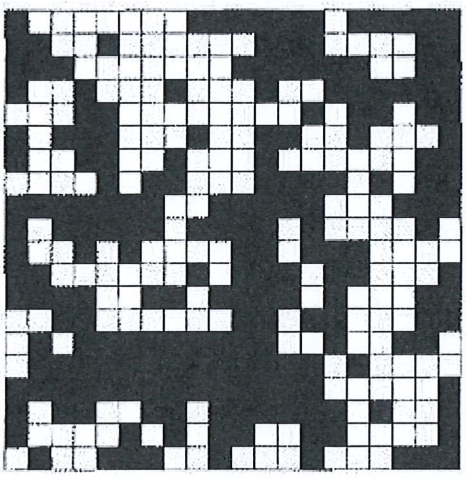
Random initial state



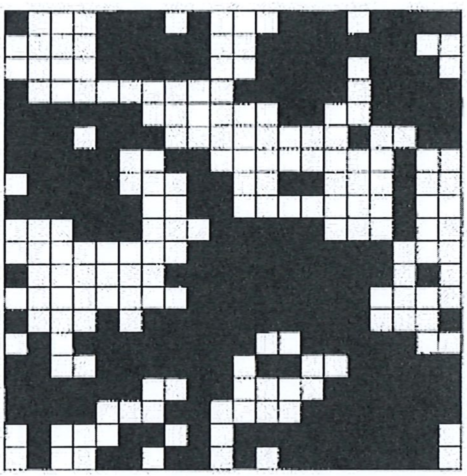
T = 10



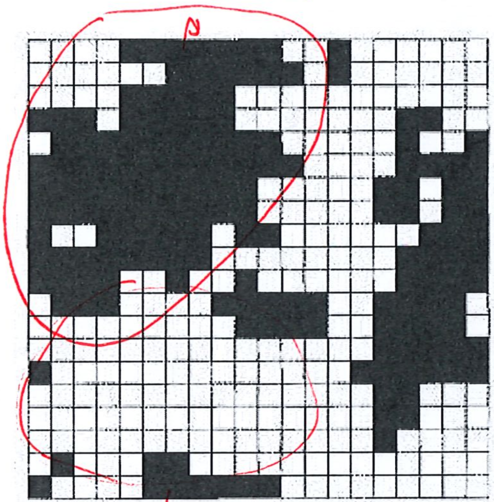
T = 5



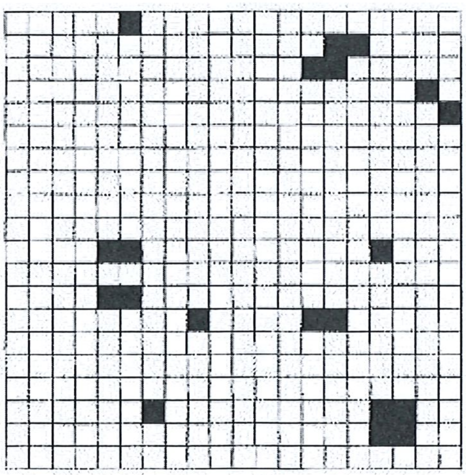
T = 4



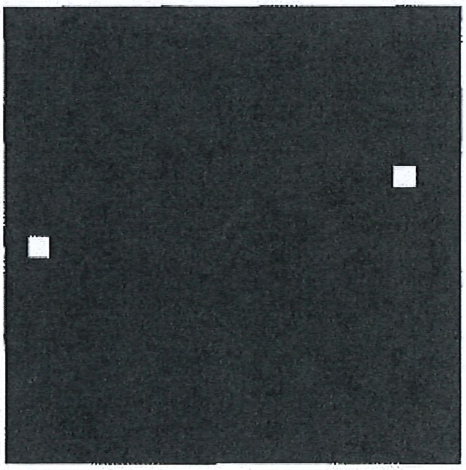
T = 3



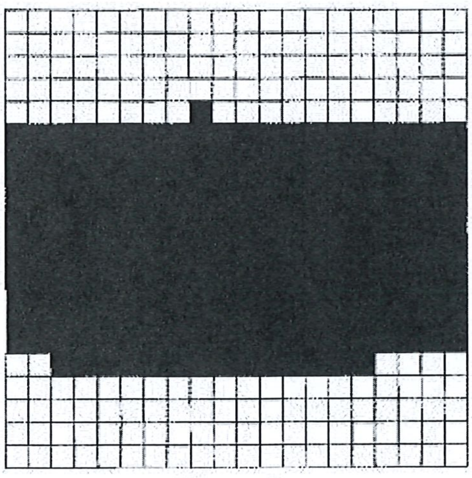
T = 2.5



T = 2



T = 1.5

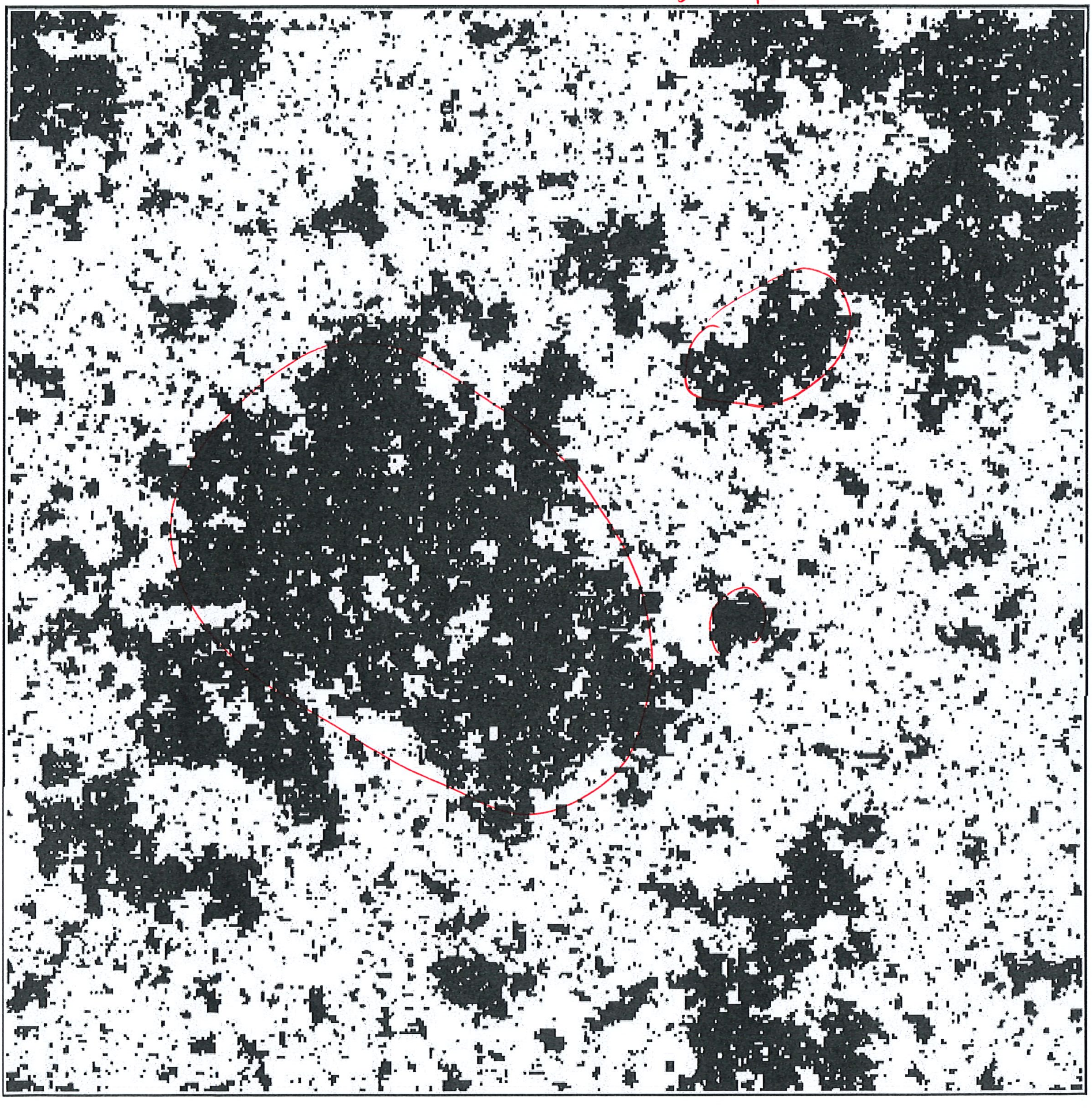


T = 1

20×20

$2^{400} \sim 10^{120}$

$\beta_c = \frac{1}{T_c} = \frac{1}{2}\log(1+\sqrt{2}) \simeq 0.44$



$$E \approx -\sum_{(jk)} \left[(s_j + s_k)\langle m \rangle + \langle m \rangle^2 \right] - H \sum_n s_n$$

each spin appear $2d$ times in sum over Nd links

$$E_{MF} = -Nd \cdot \langle m \rangle^2 - (2d\langle m \rangle + H) \sum_n s_n$$ is mean-field approx.

$$H_{eff} = 2d\langle m \rangle + H$$

Effective mag. field averaging over $2d$ n.n. of s_n

What happens to E_{MF} upon flipping $s_j \to -s_j$

$$E_{MF} = -d \cdot N\langle m \rangle^2 - H_{eff}\Big(s_j + \sum_{k \neq j} s_k\Big)$$
$$\to -d \cdot N\langle m \rangle^2 - H_{eff}\Big(-s_j + \sum_{k \neq j} s_k\Big)$$
$$\Delta E_j = 2 H_{eff} s_j$$

Independent of s_k for $k \neq j$ $\to$ non-interacting!

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Remnant of interactions in H_{eff}

Canonical partition func.

Factorizes!

$$Z_{MF} = \sum_{\{s_n\}} \exp\Big[\beta d \cdot N\langle m \rangle^2 + \beta H_{eff} \sum_n s_n\Big]$$
$$= \exp\left(\beta d \cdot N\langle m \rangle^2\right)\Big(\sum_{s_1 = \pm 1} e^{\beta H_{eff} s_1}\Big) \cdots \Big(\sum_{s_N = \pm 1} e^{\beta H_{eff} s_N}\Big)$$
$$= C\left(2\cosh(\beta H_{eff})\right)^N = C\left(2\cosh\left[\beta(2d\langle m \rangle + H)\right]\right)^N$$

$\propto \frac{\partial}{\partial H} \log Z$

Demand
$$\langle m \rangle = \frac{1}{N\beta} \frac{\partial}{\partial H} \log Z_{MF}$$
$$= \frac{\cancel{\beta}}{\cancel{\beta}} \frac{\sinh(\beta(2d\langle m \rangle + H))}{\cosh(\beta(2d\langle m \rangle + H))} = \tanh(\beta(2d\langle m \rangle + H))$$

$\to$ self-consistency condition for mean-field approx.

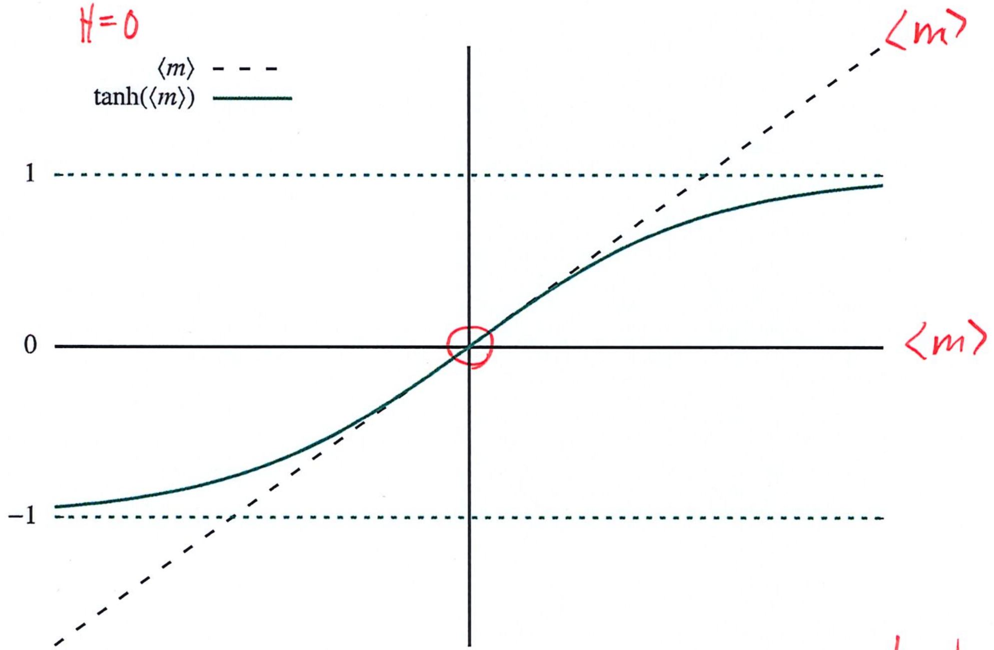
Plot $\langle m \rangle$ and $\tanh(\beta(2d \cdot \langle m \rangle + H))$ vs. $\langle m \rangle$ to find intersections

$d = 2 \quad H = 0 \quad T = 4 \to$ disordered phase, $\langle m \rangle = 0$

$\tanh(\beta(2d\langle m\rangle + H)) \rightarrow \tanh\langle m\rangle$

$d=2 \quad T=4 \rightarrow \beta = \frac{1}{4}$

$H=0$



Intersection at $\langle m\rangle = 0 \rightarrow$ disordered phase

Turn on $H \neq 0 \rightarrow$ shifts tanh

Intersection at $\langle m \rangle = \pm m_0 \neq 0$

$(H = \pm 2 \rightarrow m_0 \approx 0.88)$

$\rightarrow$ ordered phase, aligned w/mag. field

Reduce temperature $\rightarrow$ larger β in $\tanh(\beta(2d\langle m \rangle + H))$

$\rightarrow$ Faster change in tanh

$\rightarrow \langle m \rangle \approx \pm 1$ (max. magnitude)

Expect low-T ordered phase even with $H=0$

Compare $T = 2, 4, 8$ $\quad$ $\langle m \rangle = 0$ always possible

Lower-T $\rightarrow \langle |m| \rangle = m_0 \neq 0$ in additional

$m_0 \rightarrow 1$ as $T \rightarrow 0$

Imagine perturbing $\langle m \rangle = 0 \rightarrow \langle m \rangle = \varepsilon > 0$

$T=8 \rightarrow \langle m \rangle$ too large compared to tanh

$\rightarrow$ return to stable $\langle m \rangle = 0$ for equilibrium.

$T=2 \rightarrow \langle m \rangle$ too small compared to tanh

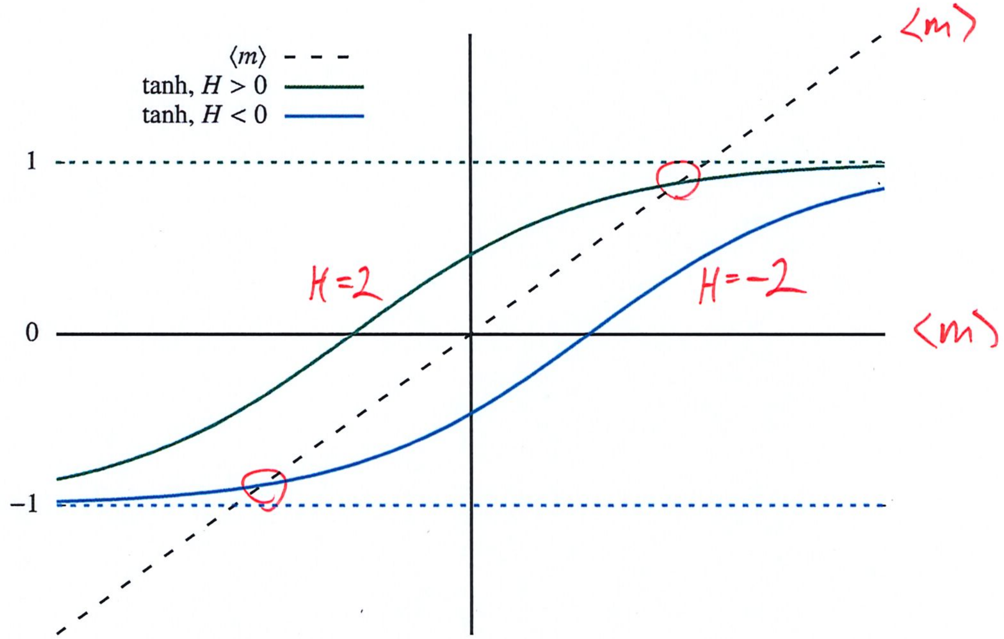
$\rightarrow$ keep going to larger $\langle m \rangle$ for equilibrium

$\hookrightarrow$ at m_0

Similarly go to $-m_0$ from $-\varepsilon < 0$

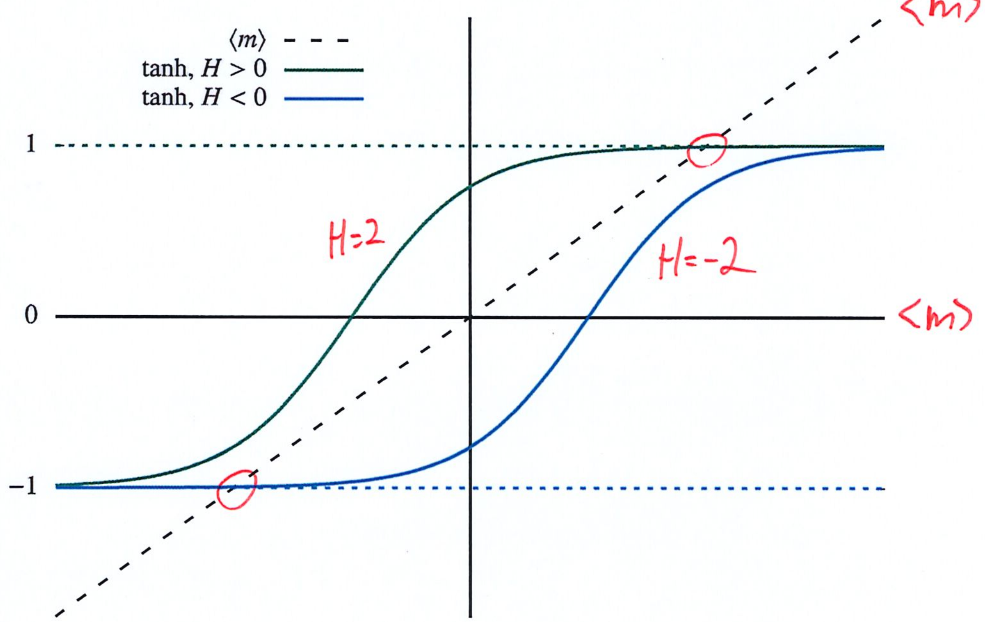
$\langle m \rangle = 0$ unstable $\rightarrow$ ordered phase at low-T

$\langle m \rangle \neq 0$



⟨m⟩
tanh, H > 0
tanh, H < 0
1
0
−1
H=2
H=−2
⟨m⟩
⟨m⟩

$J=2 \qquad T=2 \rightarrow \beta=\frac{1}{2}$



$d=2 \quad H=0$