Supplement: Relativistic ideal fermions gas (8 May)

We have seen:

Ultra-rel. boson (photon) gas

Non-rel. Fermion gas

How does an ultra-rel. fermion gas behave?

└ example: neutrinos, mass $\lesssim 10^{-37}$ kg $\sim \frac{m_e}{10^6}$

$$E_\nu = c p_\nu = \hbar \omega$$

$$\omega = \frac{2\pi c}{\lambda} = c \frac{\pi}{L} k$$

$k_{x,y,z} = 1, 2, 3, \ldots$

Easy to create & absorb

→ grand-canonical ensemble w/ $\mu = 0$

Follow photon gas steps w/ different signs (2 spins vs. 2 pol.)

$$\Phi_\nu = -\frac{VT}{c^3 \pi^2} \int_0^\infty \omega^2 \log\left(1 + e^{-\beta\hbar\omega}\right) d\omega$$

Average particle #:

$$\langle N \rangle_\nu = \frac{-\partial}{\partial \mu} \Phi_\nu \Big|_{\mu=0} = \frac{VT}{c^3\pi^2} \int_0^\infty \omega^2 \frac{\partial}{\partial \mu} \log\left(1 + e^{-\beta\hbar\omega} e^{\beta\mu}\right) d\omega \Big|_{\mu=0}$$

$$= \frac{V\cancel{T}}{c^3\pi^2} \int_0^\infty \frac{\omega^2 e^{-\beta\hbar\omega} \cancel{\beta}}{1 + e^{-\beta\hbar\omega}} d\omega \qquad x = \beta\hbar\omega = \frac{\hbar\omega}{T}$$

$$= \frac{V}{c^3\pi^2} \left(\frac{T}{\hbar}\right)^3 \int_0^\infty \frac{x^2}{e^x + 1} dx$$

└ $\left(1 - \frac{1}{2^2}\right) \Gamma(3) \zeta(3)$

$$\langle N \rangle_\nu = \frac{3}{2} \zeta(3) \left(\frac{1}{\pi^2 \hbar^3 c^3}\right) V T^3$$

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For internal energy, add factor of $\epsilon = \hbar\omega = xT$

$$\langle E\rangle_\nu = \frac{VT^4}{\pi^2\hbar^3c^3}\underbrace{\int_0^\infty \frac{x^3}{e^x+1}dx}_{\left(1-\frac{1}{2^3}\right)\Gamma(4)\zeta(4)} = 7\left(\frac{3}{4}\right)\left(\frac{\pi^4}{90}\right)\left(\frac{1}{\pi^2\hbar^3c^3}\right)VT^4 \propto \langle N\rangle_\nu T$$

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$$\langle E\rangle_\nu = 7\left(\frac{3}{4}\right)\left(\frac{2}{3\zeta(3)}\right)\left(\frac{\pi^4}{90}\right)\langle N\rangle_\nu T = \frac{7\pi^4}{180\,\zeta(3)}\langle N\rangle_\nu T$$

$$= \frac{7}{2}\frac{\zeta(4)}{\zeta(3)}\langle N\rangle_\nu T$$

Entropy $S_\nu = \dfrac{\langle E\rangle_\nu - \Phi_\nu}{T} \propto VT^3$

$$\frac{7}{4}\zeta(4) = \frac{7\pi^4}{360}$$

$$\Phi_\nu = \frac{-VT}{c^3\pi^2}\left(\frac{T}{\hbar}\right)^3\int_0^\infty x^2\log\left(1+e^{-x}\right)dx \propto VT^4$$

Condition of constant entropy $T = bV^{-1/3}$

Pressure $P_\nu = -\dfrac{\partial}{\partial V}\langle E\rangle_\nu\Big|_{S_\nu} = \dfrac{-7\pi^2}{120\hbar^3c^3}\dfrac{\partial}{\partial V}\left(b^4V^{-1/3}\right)$

$$= \frac{1}{3V}\left(\frac{7\pi^2}{120\hbar^3c^3}b^4V^{-1/3}\right) = \frac{1}{3}\frac{\langle E\rangle_\nu}{V}$$

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Equation of state

$$P_\nu V = \frac{1}{3}\langle E\rangle_\nu = \frac{7}{6}\frac{\zeta(4)}{\zeta(3)}\langle N\rangle_\nu T = \frac{7\pi^4}{540\,\zeta(3)}\langle N\rangle_\nu T \approx 1.05\langle N\rangle_\nu T$$

Very similar to photon gas, just extra factor of $\frac{7}{6}$