MATH327: Statistical Physics

Tuesday, 2 May 2023

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Something to consider

In homework question 1 you should see different high- and low-temperature behaviour even for a non-interacting spin system.

What distinguishes that situation from a true *phase transition* between distinct phases?

Simple interacting spin system on d-dim'l cubic lattice

Ising model $E_i = -\sum_{(jk)} s_j s_k - H \sum_n s_n$

(jk): $d \cdot N$ links (nn. pairs)

n: N sites

No exact solution for $3 \le d < \infty$

Consider high- and low-T limits

For simplicity set $H = 0$

$$E_i = -\sum_{(jk)} s_j s_k \qquad Z(\beta, N) = \sum_{\{s_n\}} \exp\left(\beta \sum_{(jk)} s_j s_k\right)$$

$$\boxed{\text{High-T } \beta \to 0 \text{ limit: } \quad Z \to \sum_{\{s_n\}} \exp(0) = 2^N}$$

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Equal $p_i = 1/2^N$ for all micro-states

E_i negligible compared to T

Characterize system by magnetization $m = \frac{n_+ - n_-}{n_+ + n_-}$ $\quad -1 \leq m \leq 1$

$N = n_+ + n_-$

$H=0 \rightarrow$ symmetry under flipping all spins
$\pm 1 \rightarrow \mp 1$

$\rightarrow$ consider ~~b~~ $0 \leq |m| \leq 1$

$$\langle |m| \rangle = \sum_i |m|_i \, P_i$$

$T \to \infty$ all micro-states equally probable
just need to count them

$$\binom{N}{n_+} = \binom{N}{n_-} = \frac{N!}{n_+! \, n_-!} = \frac{(n_+ + n_-)!}{n_+! \, n_-!}$$

$N \gg 1 \rightarrow$ sharp peak at $n_+ = n_- = \frac{1}{2} N \rightarrow |m| = 0$

$N \to \infty$ thermodynamic limit gives $\langle |m| \rangle \to 0$

"disordered phase"

For low-T $\beta \to \infty$

As in non-interacting case, min-energy ground state exponentially more probable

$P_i \sim e^{-\beta E_i}$

$H=0 \rightarrow$ 2 degenerate ground states

$(n_+, n_-) = (N, 0)$ & $(0, N) \rightarrow |m| = 1$

$$\boxed{E_0 = -\sum_{(jk)} s_j s_k = -\sum_{(jk)} (\pm 1)^2 = -d \cdot N} \quad (-1 \text{ from every link})$$

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First excited energy level flips single spin (assume $d > 1$)

$(n_+, n_-) = (N-1, 1)$ and $(1, N-1)$

$N + N = 2N$ degenerate micro-states

All with $|m| = \frac{N-2}{N} = 1 - \frac{2}{N}$

Prob depend on

$E_1 = 2d - (d \cdot N - 2d) = -(d \cdot N - 4d)$

+1 from 2d links connected to flipped spin

-1 from other $d \cdot N - 2d$ links

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Lower prob + higher degeneracy — which wins?

$$\frac{P(E_0)}{P(E_1)} = \frac{2\exp(\beta \cancel{d \cdot N})}{2N\exp(\beta(\cancel{d \cdot N} - 4d))} = \frac{\exp(4\beta d)}{N}$$

$T \to 0$ $\beta \to \infty$ $\to$ exponentially approach $\langle |m| \rangle \to 1$ "ordered phase"

Magnetization is Ising model order parameter (OP)

distinguish $\langle |m| \rangle \to 0$ disordered phase (high-T)

from $\langle |m| \rangle \to 1$ ordered phase (low-T)

In general, zero vs. non-zero order parameter (as $N \to \infty$) distinguishes phases

If OP or its derivative(s) discontinuous or divergent then phase transition at critical point of control params.

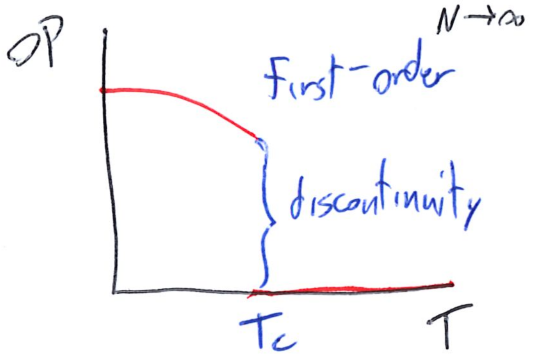
Otherwise crossover rather than true transition

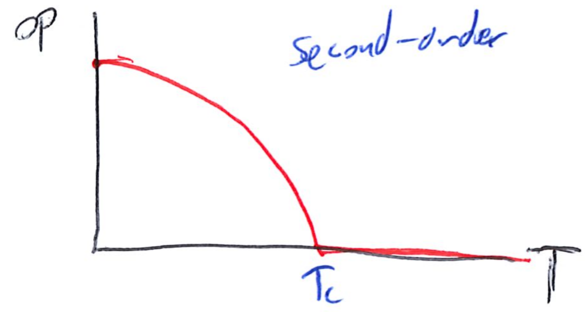
$H=0 \to$ Ising critical point defined by critical temperature T_c

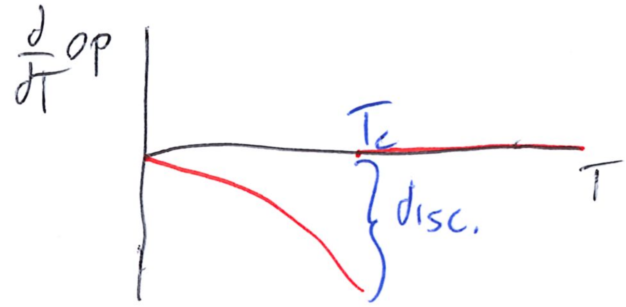
Formally, OP related to derivative of free energy

Discontinuous OP $\to$ "first-order" transition

Continuous OP w/discontinuous or divergent derivative $\to$ "second-order" transition







Relate $\langle m \rangle$ to derivative of $F = -T \log Z$

Restore magnetic field

$$E_i = -\sum_{(jk)} s_j s_k - H \sum_n s_n = -\sum_{(jk)} s_j s_k - HNm$$

$$m = \frac{1}{N}(n_+ - n_-) = \frac{1}{N}\sum_n s_n$$

So $Z = \sum_{\{s_n\}} \exp\left(\beta \sum_{(jk)} s_j s_k + \beta HNm\right)$

Consider $\frac{\partial F}{\partial H} = -T \frac{1}{Z}\frac{\partial Z}{\partial H} = -\cancel{T}\frac{1}{Z}\sum_{\{s_n\}} \cancel{\beta} Nm \exp\left(\beta \sum_{(jk)} s_j s_k + \beta HNm\right)$

$$= -N\langle m \rangle$$

$$\therefore \langle m \rangle = \frac{1}{N\beta}\frac{\partial}{\partial H}\log Z$$

as promised for order param.

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