Thu 27 Apr

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Einstein solid micro-states for N oscillators

K units of energy

$$M = \frac{(K+N-1)!}{K!\,(N-1)!} = \binom{K+N-1}{K}$$

What not to do:

$$\frac{N^K}{\text{over-counting}} \to \text{mess}$$

Trick:

Putting K indist'able balls into N dist'able boxes

Count # of sequences $\bullet\bullet|\bullet||\bullet \to (2,1,0,1)$

$K=4,\ N=4$

Total $K+N-1$ symbols in sequence

Choose K of them to be balls $\to M = \binom{K+N-1}{K}$

Check: $N=3$, $K=0$: $\binom{2}{0}=1$ ✓ $K=1$: $\binom{3}{1}=3$ ✓ $K=2$: $\binom{4}{2}=\frac{4!}{2\cdot 2}=6$ ✓ $K=3$: $\binom{5}{3}=\frac{5!}{3!\cdot 2!}=10$ ✓

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{1}{\hbar\omega}\frac{\partial}{\partial K}\log\left(\frac{(K+N-1)!}{K!\,(N-1)!}\right) \to \frac{1}{\hbar\omega}\log\left(1+\frac{N}{K}\right)$$

$E = K\hbar\omega$

$N-1 \approx N$

$N \gg 1$

$$\log\left(1+\frac{N\hbar\omega}{E}\right) = \beta\hbar\omega$$

$$E = \frac{N\hbar\omega}{e^{\beta\hbar\omega}-1}$$

$$c_V = \frac{\partial E}{\partial T} = -\beta^2\frac{\partial E}{\partial\beta} = \frac{N\hbar^2\omega^2 e^{\beta\hbar\omega}\beta^2}{(e^{\beta\hbar\omega}-1)^2} = \frac{Nx^2e^x}{(e^x-1)^2}$$

$x = \beta\hbar\omega$

Check low- and high-T limit

$T \to 0 \quad \beta \to \infty \quad x \to \infty \qquad x = \beta\hbar\omega = \hbar\omega/T$

$$\frac{C_V}{N} = \frac{x^2 e^x}{(e^x-1)^2} \approx \frac{x^2 e^x}{e^{2x}} = \frac{x^2}{e^x} \to 0 \checkmark$$

(third law)

$T \to \infty \quad \beta \to 0 \quad x \to 0$

Expand <u>carefully</u> in case of cancellations

$$\frac{C_V}{N} = \frac{x^2\left(1 + x + \frac{1}{2}x^2 + \dots\right)}{\left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots\right)^2} = \frac{1 + x + \frac{1}{2}x^2 + \dots}{\left(1 + \frac{1}{2}x + \frac{1}{6}x^2 + \dots\right)^2}$$

$$= \frac{1 + x + \frac{1}{2}x^2 + \dots}{1 + \left[x + x^2\left(\frac{1}{4} + \frac{1}{3}\right) + \dots\right]}$$

$$= \left(1 + x + \frac{1}{2}x^2\right)\left[1 - \left(x + \frac{7}{12}x^2\right) + \left(x + \frac{7}{12}x^2\right)^2\right] + O(x^3)$$

$$= 1 + x(1-1) + x^2\left(\frac{1}{2} \cancel{-1} - \frac{7}{12}x^2 \cancel{+1}\right) + O(x^3)$$

$$= 1 - \frac{1}{12}\left(\frac{\hbar\omega}{T}\right)^2 + O\left(\frac{\hbar^3\omega^3}{T^3}\right)$$

Low-T experiments

$$\frac{C_V}{T} = \alpha + \gamma T^2 \to C_V = \alpha T + \gamma T^3$$

Issue w/ Einstein solid at low T

Oscillations correlated among multiple atoms

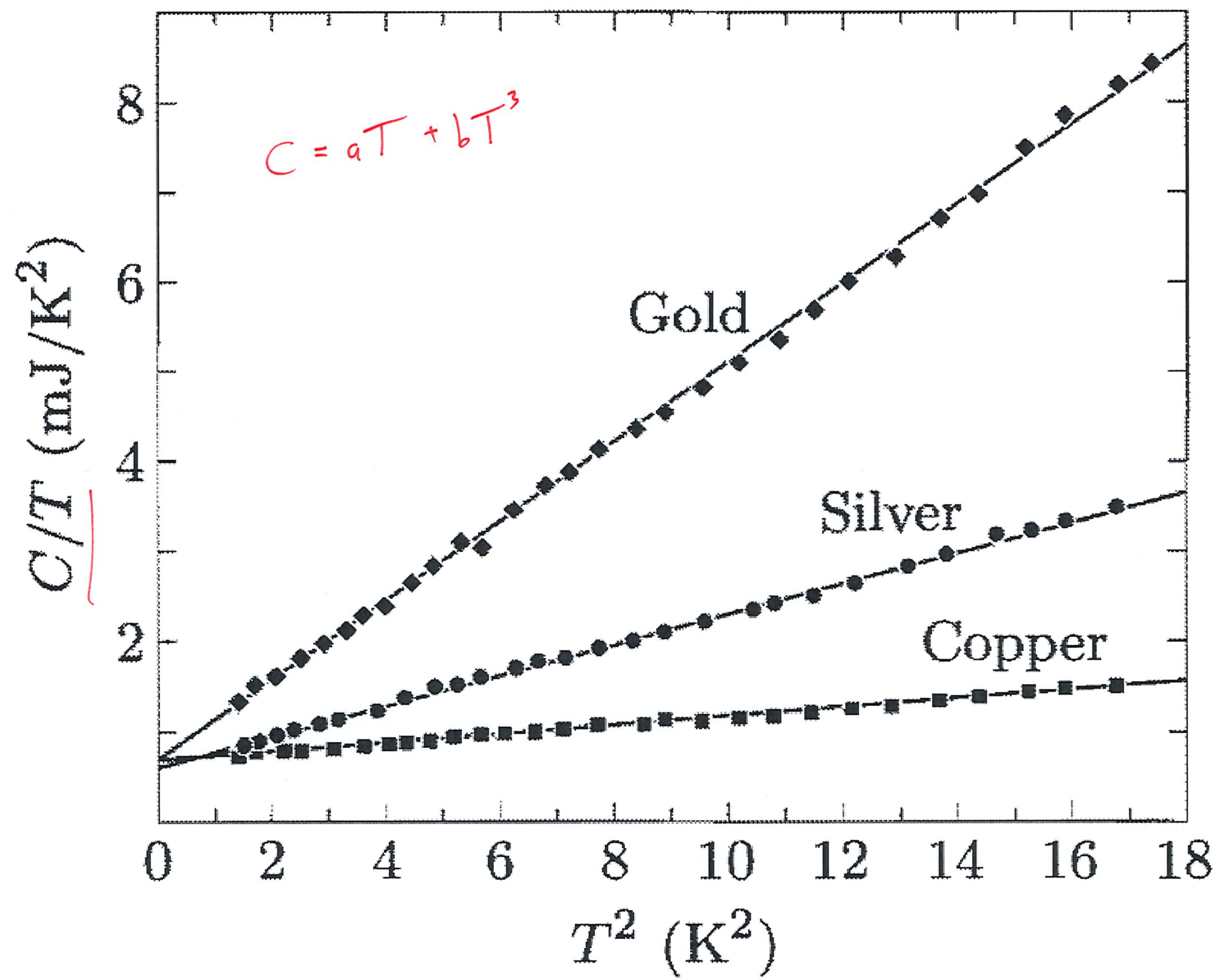
Ultimately $\to$ propagating waves of coherent motion

$\hookrightarrow$ phonons inspired by photons

Phonons are massless bosons ($E = \hbar\omega$)

travelling at the speed of sound $c_s \ll c$

Maximum ~~wavelength~~ frequency $\leftrightarrow$ minimum wavelength



C = aT + bT^3
Gold
Silver
Copper
C/T (mJ/K^2)
T^2 (K^2)
0
2
4
6
8
10
12
14
16
18

Task: Redo photon gas for phonons $\omega_{max} = T/T_0$
to find high-T and low-T c_v function form
Einstein T^3

Very low temperature → atoms frozen in place
(not enough energy for phonons)

Electron gas → non-zero $\langle E \rangle_f$ as $T \to 0$
→ ~~non-zero~~ low-T c_v

Task: Need to go beyond step func. approx. of $F(E)$
Integrate $\langle E \rangle_f \propto \int_0^\infty F(E)\, E^{3/2}\, dE$ by parts
Argue boundary term vanishes
remaining integrand sharply peaked around $E \approx \mu$
Taylor expand $E^{5/2}$ around $E \approx \mu$
→ simpler integrals → $c_v = \frac{\partial}{\partial T} \langle E \rangle_f$