Tue 18 Apr

43 49 11

Photon gas from grand-canonical ensemble and quantum Bose-Einstein statistics

$$\frac{\langle E \rangle_{ph}}{V} = \int_0^\infty P(\omega)\, d\omega = \int_0^\infty P(\lambda)\, d\lambda \qquad c = \frac{\lambda \omega}{2\pi}$$

Planck spectrum $\quad P(\omega) = \left(\frac{\hbar}{c^3 \pi^2}\right) \frac{\omega^3}{e^{\beta \hbar \omega} - 1}$

$$P(\lambda) = \left(\frac{16 \pi^2 \hbar c}{\lambda^5}\right) \frac{1}{e^{2\pi \beta \hbar c/\lambda} - 1}$$

Good description of sun, stars, cosmic microwave background

Finish integrating

$$\langle E \rangle_{ph} = \frac{V\hbar}{c^3 \pi^2} \int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega} - 1}\, d\omega = \frac{V\hbar}{c^3\pi^2} \left(\frac{T}{\hbar}\right)^4 \underbrace{\int_0^\infty \frac{x^3}{e^x - 1}\, dx}_{\Gamma(4)\zeta(4) = 6 \cdot \frac{\pi^4}{90}}$$

$$x = \beta\hbar\omega = \frac{\hbar\omega}{T}$$

$$d\omega = \frac{T}{\hbar}\, dx$$

$$\boxed{\langle E \rangle_{ph} = \frac{\pi^2}{15 \hbar^3 c^3} V T^4}$$

page 110

Compare w/ classical, canonical, non-rel ideal gas $\langle E \rangle = \frac{3}{2} N T$

$\to$ compute grand-canonical $\langle N \rangle_{ph} = \frac{-\partial}{\partial \mu} \Phi_{ph} \Big|_{\mu=0}$

$$\langle N \rangle_{ph} = \frac{-VT}{c^3\pi^2} \int_0^\infty \omega^2 \frac{\partial}{\partial \mu} \log\left(1 - e^{-\beta\hbar\omega} e^{\beta\mu}\right) d\omega \Big|_{\mu=0}$$

$$= \frac{+V\cancel{T}}{c^3\pi^2} \int_0^\infty \frac{\omega^2 (\cancel{-} e^{-\beta\hbar\omega} e^{\beta\mu})(\cancel{\beta})}{1 - e^{-\beta\hbar\omega} e^{\beta\mu}}\, d\omega \Big|_{\mu=0}$$

$$= \frac{V}{c^3\pi^2} \int_0^\infty \frac{\omega^2}{e^{\beta\hbar\omega} - 1}\, d\omega = \frac{V}{c^3\pi^2} \left(\frac{T}{\hbar}\right)^3 \underbrace{\int_0^\infty \frac{x^2}{e^x - 1}\, dx}_{\Gamma(3)\zeta(3)}$$

$$= \frac{2\zeta(3)}{\pi^2 \hbar^3 c^3} V T^3$$

page 111

As before, $\langle E\rangle_{ph} \propto \langle N\rangle_{ph} T$

constant factor $\left(\frac{\pi^2}{15\hbar^3c^3}\right)\left(\frac{\pi^2\hbar^3c^3}{2\zeta(3)}\right)$

$$= \frac{6\pi^4/90}{2\zeta(3)} = \frac{\Gamma(4)\zeta(4)}{\Gamma(3)\cancel{3}\zeta(3)} \simeq 2.7$$

Radiation pressure

$$P_{ph} = \frac{-\partial}{\partial V}\langle E\rangle_{ph}\Big|_{S_{ph}}$$

Need constant entropy $S_{ph} = \dfrac{\langle E\rangle_{ph} - \Phi_{ph}}{T}$

$$\frac{\Phi_{ph}}{T} = \frac{V}{c^3\pi^2}\int_0^\infty \omega^2 \log\left(1-e^{-\beta\hbar\omega}\right) d\omega = \frac{VT^3}{\pi^2\hbar^3c^3}\underbrace{\int_0^\infty x^2 \log\left(1-e^{-x}\right)dx}_{-2\zeta(4) = \frac{-\pi^4}{45}}$$

$$S_{ph} = VT^3\left(\frac{\pi^2}{\hbar^3c^3}\right)\left(\frac{1}{15}+\frac{1}{45}\right) = \underbrace{VT^3}_{\text{constant}}\left(\frac{4\pi^2}{45\hbar^3c^3}\right) \qquad T = bV^{-1/3}$$

$$\boxed{\begin{aligned} P_{ph} &= \frac{-\pi^2}{15\hbar^3c^3}\frac{\partial}{\partial V}\left(b^4V^{-1/3}\right) \\ &= \frac{1}{3V}\left(\frac{\pi^2}{15\hbar^3c^3}\right)\left(b^4V^{-1/3}\right) = \frac{1}{3V}\langle E\rangle_{ph} \end{aligned}}$$

page 112

$$P_{ph}V = \frac{1}{3}\langle E\rangle_{ph} = \left(\frac{\Gamma(3)}{\Gamma(4)}\right)\left(\frac{\Gamma(4)\zeta(4)}{\Gamma(3)\,\zeta(3)}\right)\langle N\rangle_{ph}T = \underbrace{\frac{\pi^4}{90\zeta(3)}}_{\simeq 0.9004}\langle N\rangle_{ph}T$$

Photon gas equation of state
similar to ideal gas law

Ideal fermions gas

$$n_l = 0, 1 \longrightarrow \Phi_F = -T\sum_{l=0}^{\infty}\log\left(1+e^{-\beta(E_l-\mu)}\right)$$

Back to non-rel. $E = \dfrac{\hbar^2\pi^2}{2mL^2}\underbrace{\left(k_x^2+k_y^2+k_z^2\right)}_{k^2}$ $\qquad k_{x,y,z} = 1, 2, \dots$

$$\Phi_F = -2T \sum_{\vec{k}} \log\left[1 + \exp\left(-\frac{\hbar^2 \pi^2 \vec{k}^2}{2mL^2 T} + \frac{\mu}{T}\right)\right]$$

two "spin" states per $\vec{k}$

Same simplification: Integrate over (positive) momenta in spherical coords

$$\Phi_F \approx -\pi T \int_0^\infty \hat{k}^2 \log\left[1 + \exp\left(\frac{-\hbar^2 \pi^2 \hat{k}^2}{2mL^2 T} + \frac{\mu}{T}\right)\right] d\hat{k}$$

Change variables to energy $\quad \hat{k} = \frac{L\sqrt{m}}{\pi\hbar}\sqrt{2E}$

$$d\hat{k} = \frac{L\sqrt{m}}{\pi\hbar}\frac{dE}{\sqrt{2E}}$$

$$\boxed{\begin{aligned}\Phi_F &\approx -\pi T \left(\frac{L\sqrt{m}}{\pi\hbar}\right)^3 \int_0^\infty (2E) \log\left(1 + e^{-\beta(E-\mu)}\right) \frac{dE}{\sqrt{2E}} \\ &\approx -VT \frac{\sqrt{2m^3}}{\pi^2\hbar^3} \int_0^\infty \log\left(1 + e^{-\beta(E-\mu)}\right) \sqrt{E}\, dE\end{aligned}}$$

page 114

Next simplification: Considering low temperatures