Thu 23 Mar

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Plan

Homework

Cycles – Otto vs. Carnot & Diesel

Einstein solid

HW1

CLT gives prob. distribution $p(x)$

→ probabilities requires integrated $P(a \le X \le b) = \int_a^b p(x)\,dx$

Two snapshots of same distribution at time t

and time $t' = t + \Delta t$ (3.5 hours)

$$\frac{1}{\sqrt{2\pi t' D^2}} \int_X^\infty \exp\left[-\frac{(x - v_{dr}t')^2}{2t'D^2}\right] dx = \frac{584}{\sqrt{2\pi t D^2}} \int_X^\infty \exp\left[-\frac{(x - v_{dr}t)^2}{2tD^2}\right] dx$$

Solve for t...

$$u = \frac{x - v_{dr}t}{2tD^2}$$

$$1 - \text{erf}\left(\frac{x - v_{dr}(t+\Delta t)}{2(t+\Delta t)D^2}\right) = 584\left(1 - \text{erf}\left(\frac{x - v_{dr}t}{2tD^2}\right)\right)$$

HW2

"Heat exchange → temperatures move closer to each other" X

Temperatures move farther apart

"unnatural"

MATH327: Statistical Physics, Spring 2023
Homework 1 — Comments on Question 1

The two measurements given in the first homework question — $1\ \mathrm{GBq}$ at time t and $584\ \mathrm{GBq}$ at time $t+3.5$ hours — allow us to estimate the time of the accident as well as the amount of radioactivity released. Since they tell us the probability $P(x \geq X)$ increased by a factor of 584 during the $\Delta t = 3.5$ hours from time t to time $t' = t + \Delta t$, we just need to solve

$$\frac{1}{\sqrt{2\pi t' D^2}} \int_X^\infty \exp\left[-\frac{(x - v_{\mathrm{dr}} t')^2}{2t' D^2}\right] dx = \frac{584}{\sqrt{2\pi t D^2}} \int_X^\infty \exp\left[-\frac{(x - v_{\mathrm{dr}} t)^2}{2t D^2}\right] dx$$

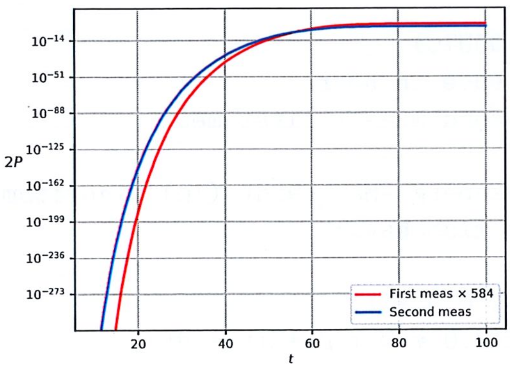
for t. This is easier said than done.

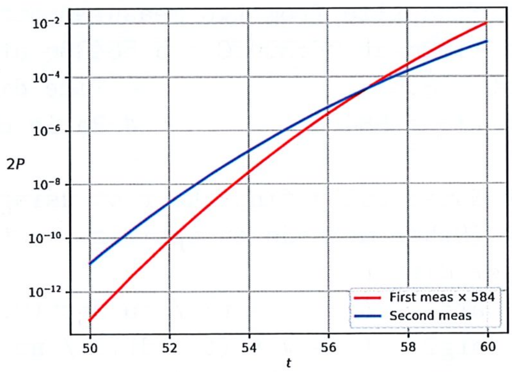
We can make some progress by relating each of these integrals to the error function, as in the model solution. Doing so converts the equation above into

$$1 - \mathrm{erf}\left(\frac{x - v_{\mathrm{dr}}(t + \Delta t)}{\sqrt{2(t + \Delta t) D^2}}\right) = 584\left[1 - \mathrm{erf}\left(\frac{x - v_{\mathrm{dr}} t}{\sqrt{2t D^2}}\right)\right].$$

While this is still not convenient to work with by hand, modern numerical methods can easily deal with it.

One strategy is to plot both sides of the equation above vs. t and see where they intersect — this intersection point occurs at the t for which the two measurements are consistent. This is done in the two plots below. The plot on the left first considers a large range $10 \leq t \leq 100$, from which we can see that the desired intersection occurs slightly before $t = 60$. Zooming in on $50 \leq t \leq 60$ in the second plot on the right, we can read off $t \approx 57$.





While we could continue zooming in, an alternative approach is to numerically find the appropriate root of the difference

$$\mathrm{erfc}\left(\frac{x - v_{\mathrm{dr}}(t + \Delta t)}{\sqrt{2(t + \Delta t) D^2}}\right) - 584\,\mathrm{erfc}\left(\frac{x - v_{\mathrm{dr}} t}{\sqrt{2t D^2}}\right) = 0,$$

where $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$. It can help the root-finding algorithm if we first estimate roughly where it should look (around $t \approx 60$). Figures like those above are a good

Otto cycle

$$W_{out} = -W_{34} = -\Delta\langle E\rangle = \frac{-3}{2}N(T_4 - T_3)$$

$$W_{in} = \frac{3}{2}N(T_2 - T_1)$$

$$Q_{in} = \frac{3}{2}N(T_3 - T_2)$$

$$\eta = \frac{T_3 - T_4 - T_2 + T_1}{T_3 - T_2} = 1 - \frac{T_4(1 - T_1/T_4)}{T_3(1 - T_2/T_3)}$$

Constant $S \propto VT^{3/2}$

$$V_2 T_3^{3/2} = V_1 T_4^{3/2} \rightarrow \frac{T_4}{T_3} = \left(\frac{V_2}{V_1}\right)^{2/3} = \frac{T_1}{T_2} = \frac{1}{r^{2/3}}$$

$$\frac{T_1}{T_4} = \frac{T_2}{T_3}$$

$$\eta = 1 - \frac{T_4}{T_3} = 1 - \frac{1}{r^{2/3}} = 1 - \frac{T_1}{T_2}$$

$$\eta_c = 1 - \frac{T_1}{T_3} > \eta = 1 - \frac{T_1}{T_2}$$

Max η for large $r = \frac{V_1}{V_2}$

Realistic $\eta \sim 20\% - 30\%$

Too large $\frac{V_1}{V_2} \rightarrow$ auto-ignite "knock"

Diesel

Compress only air, then inject (diesel) fuel

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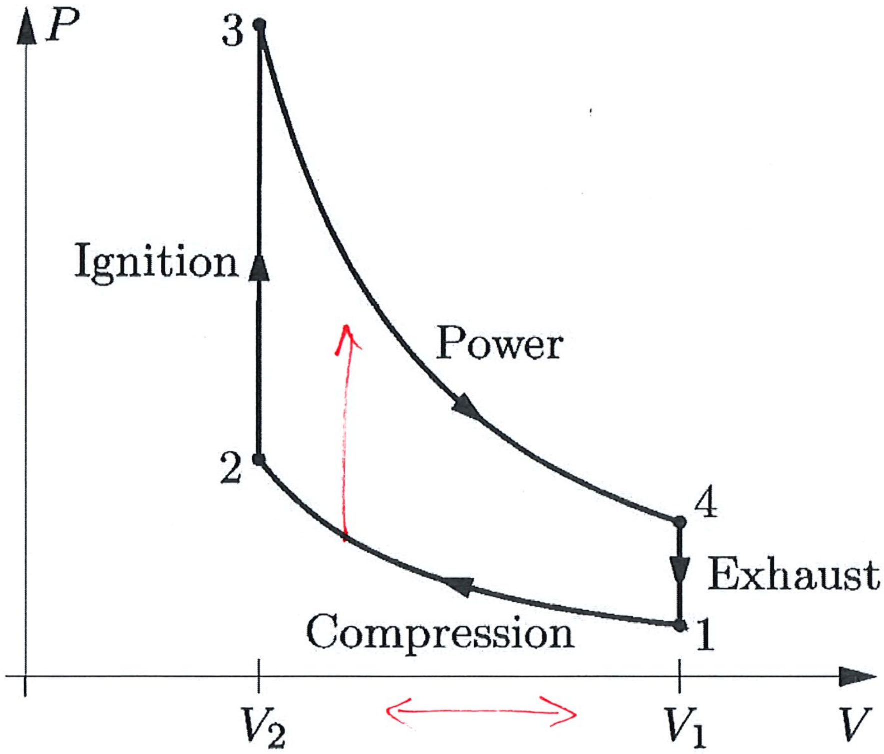
Tutorial activity — Otto cycle

The figure below shows the 'Otto cycle' that describes an idealized petrol engine.

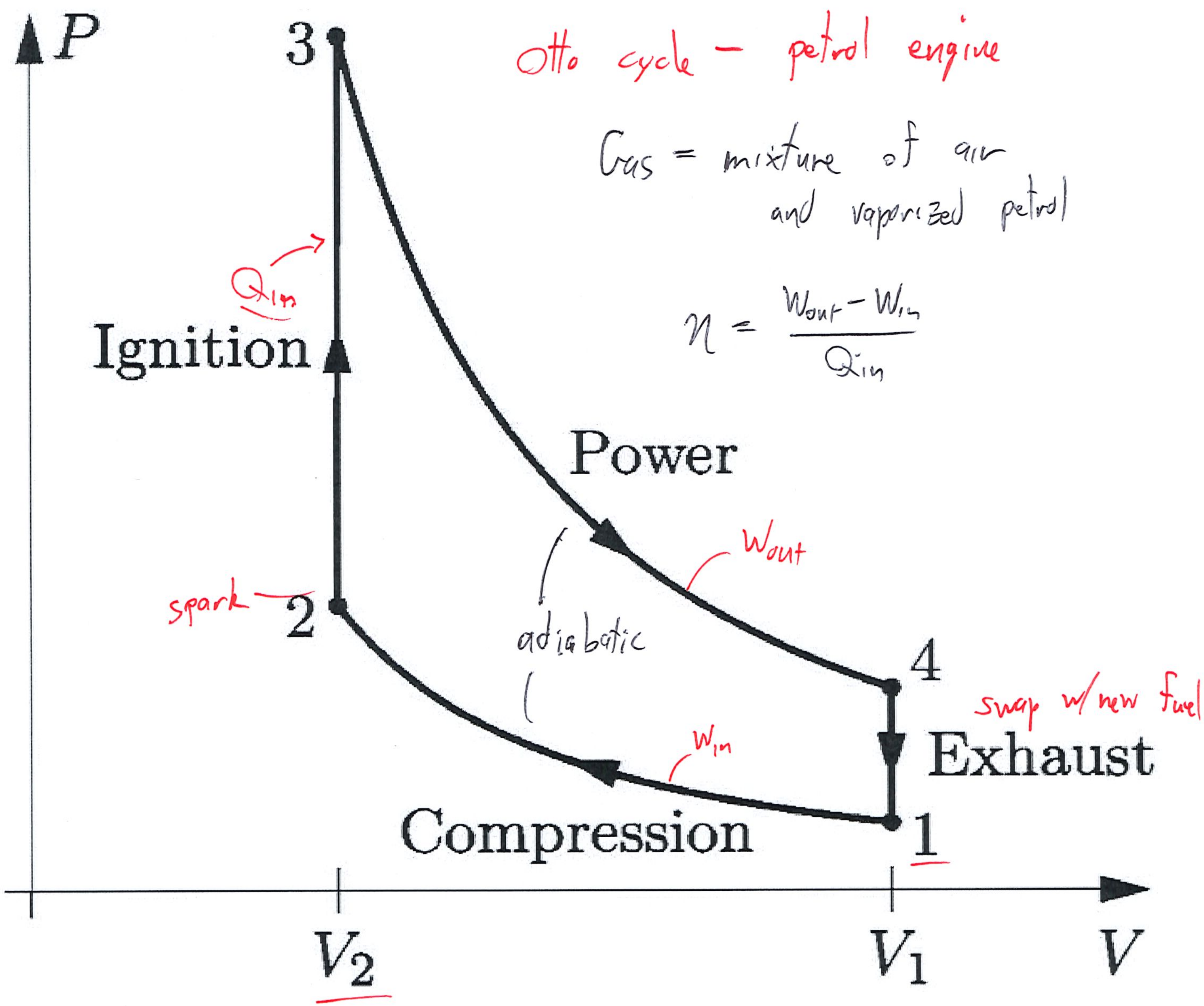
- Fast (adiabatic) compression increases the pressure of the gas (a mixture of air and vaporized petrol), until a spark ignites it.
- This ignition introduces lots of heat almost instantaneously, while the volume is fixed at V_2. Even though the gas itself is burning, we can interpret this heat as coming from energy exchange with a hot thermal reservoir.
- The gas then does work by adiabatically expanding back to volume $V_1 > V_2$.
- Finally, heat is expelled at fixed volume V_1 by swapping the hot exhaust for an equal amount of cooler, fresh gas ready to be burned.



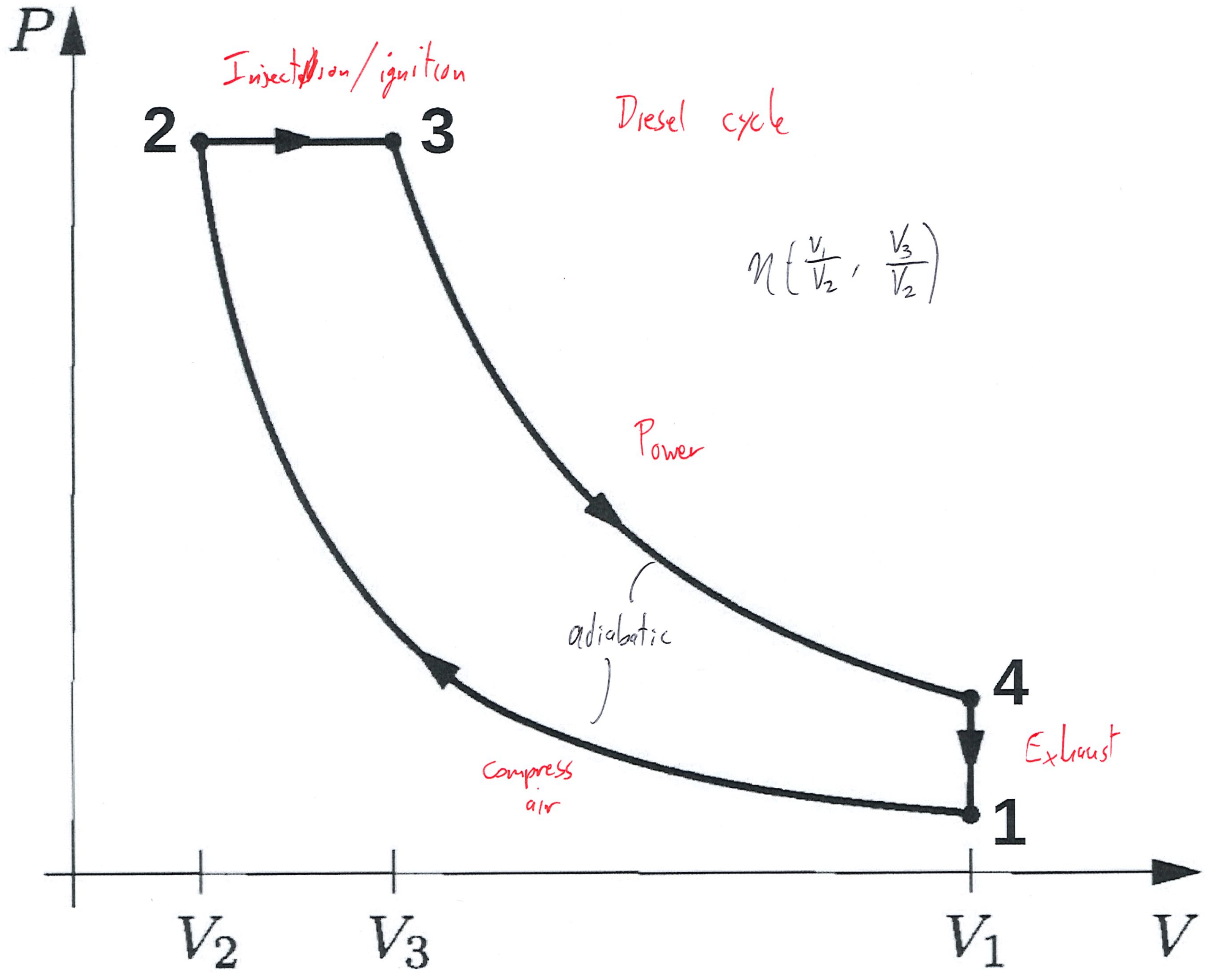
The efficiency η of the Otto cycle depends on the **compression ratio**

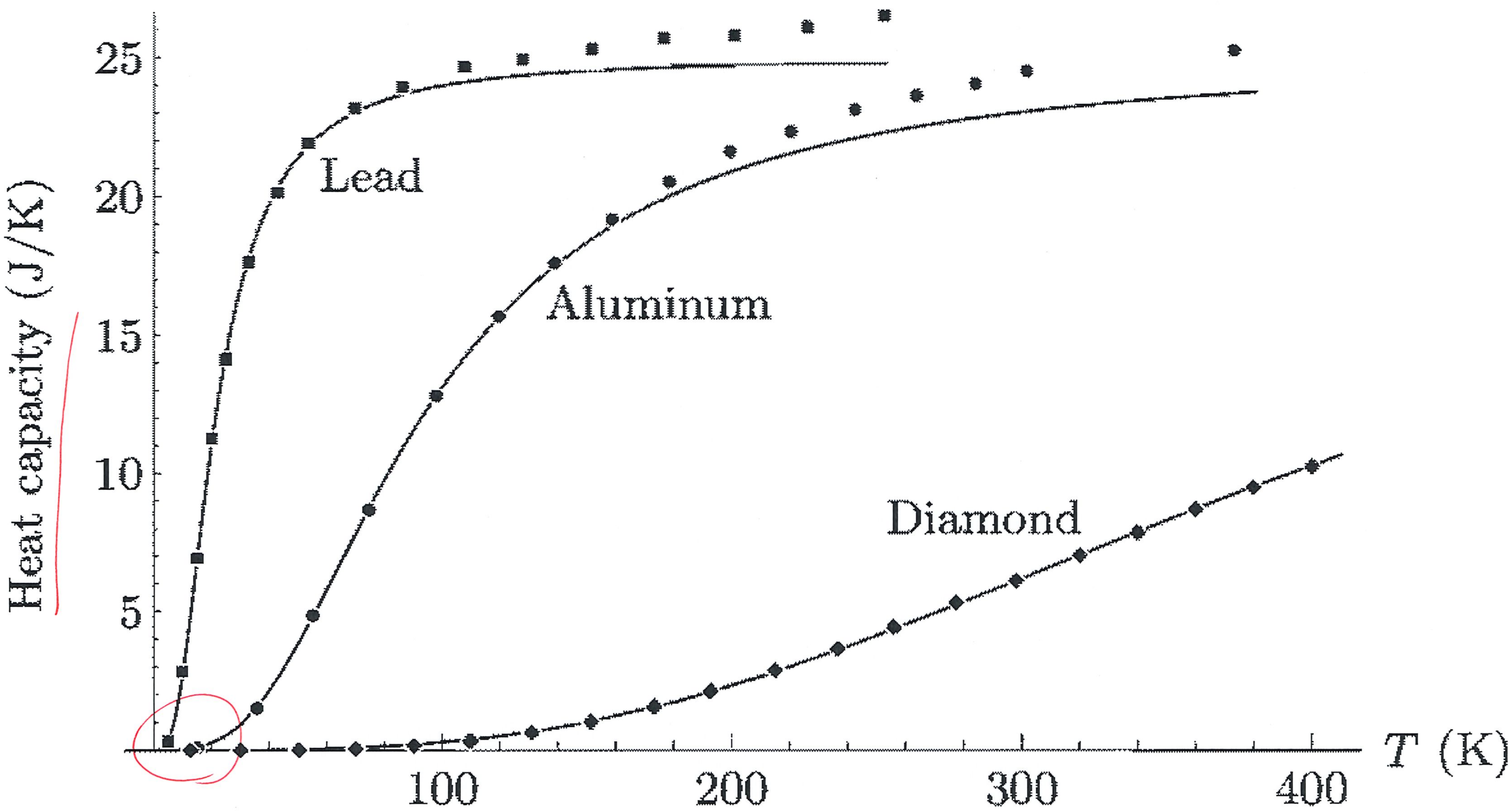
$$r \equiv \frac{V_1}{V_2} > 1.$$

What is this efficiency? How does it compare to the efficiency of the Carnot cycle? How should V_1 and V_2 be chosen to maximize the efficiency?



Otto cycle - petrol engine
Gas = mixture of air and vaporized petrol
$\eta = \frac{W_{out} - W_{in}}{Q_{in}}$
P
3
Q_{in}
Ignition
Power
W_{out}
spark
2
adiabatic
4
swap w/ new fuel
W_{in}
Exhaust
Compression
1
V_2
V_1
V


P
Injection/ignition
2
3
Diesel cycle
η(V1/V2, V3/V2)
Power
adiabatic
4
Exhaust
compress air
1
V2
V3
V1
V


Heat capacity (J/K)
25
20
15
10
5
Lead
Aluminum
Diamond
100
200
300
400
T (K)

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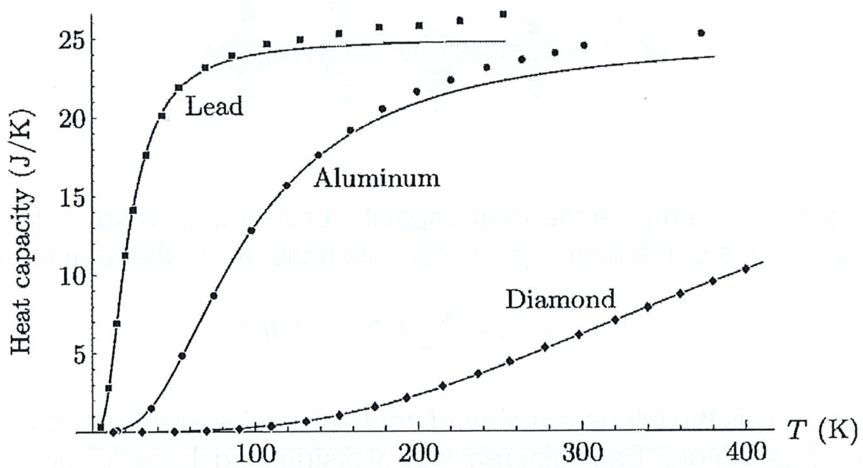
Tutorial activity — Einstein solid

In Section 3.4 we computed the energy for N distinguishable spins in a solid,

$$E = -NH\tanh(\beta H)$$

for inverse temperature $\beta = 1/T$ and magnetic field strength H. What is the corresponding heat capacity? How does it compare to the experimental[1] data points in the figure below (from Schroeder's *Introduction to Thermal Physics*)?



You should find poor agreement — especially upon turning off the external field by taking $H \to 0$! This issue turns out to persist even for more realistic models of solids analyzed using classical approaches. To address it, in 1907 Einstein developed a simple model of solids based on quantized energies, taking some inspiration from his 1905 proposal that quantized energies explain the photoelectric effect.

The 'Einstein solid' consists of many atoms whose positions are fixed to (distinguishable) locations in a regular lattice. Interactions between neighbouring atoms are credited with pinning down each atom to its fixed location. This is modeled by picturing neighbouring atoms connected by 'oscillators', analogous to springs, which possess energy as a consequence of these interactions. We define the Einstein solid by hypothesizing that the energy of each oscillator is quantized, $\varepsilon_i = 0, \hbar\omega, 2\hbar\omega, \cdots$, with the same characteristic angular frequency ω for all oscillators. Although these oscillators model interactions between nearest-neighbour atoms, in this approach they are *non-interacting* degrees of freedom that we can analyze using the statistical physics tools we have already developed.

As illustrated by the figure below, also from Schroeder's *Introduction to Thermal Physics*, the number of oscillators depends both on the number of atoms and

[1]Experimentally it is easier to measure the heat capacity at constant *pressure*, c_p, rather than at constant volume, but the difference between c_p and c_v is negligible for our purposes here.