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810573

Quantum statistics

↳ micro-states from energy levels occupation #s

Bosons $n_\ell = 0, 1, 2, \dots \longrightarrow Z_g^{BE} = \prod_\ell \frac{1}{1 - e^{-\beta(E_\ell - \mu)}}$

$(\mu < E_\ell)$

Fermions $n_\ell = 0, 1$

Fermi-Dirac statistics for fermions

Single energy level $\mathcal{E}_0 \rightarrow Z_g = \sum_{n_0=0}^{1} e^{-\beta(E_0 - \mu)n_0} = 1 + e^{-\beta(E_0 - \mu)}$

$E_i = E_0 n_0 \longrightarrow \sum_\ell E_\ell n_\ell \qquad N_i = \sum_\ell n_\ell$

General $\mathcal{E}_\ell \rightarrow Z_g^{FD} = \sum_{n_0} \sum_{n_1} \dots \sum_{n_L} \exp\left[-\beta \sum_{\ell=0}^{L} (E_\ell - \mu) n_\ell\right]$

$= \left(\sum_{n_0} e^{-\beta(E_0 - \mu)n_0}\right) \dots \left(\sum_{n_L} e^{-\beta(E_L - \mu)n_L}\right)$ (factorization)

$= \prod_\ell \left(1 + e^{-\beta(E_0 - \mu)}\right)$ no need for $\mu < E$

product like $Z_N \sim Z_1^N$
but w/out over-counting

Collect results

$\Phi_{MB}(T, \mu) = -T \log\left(\prod_\ell \exp\left[e^{-\beta(E_\ell - \mu)}\right]\right) = -T \sum_\ell e^{-\beta(E_\ell - \mu)}$

$\Phi_{BE} = T \sum_\ell \log\left(1 - e^{-\beta(E_\ell - \mu)}\right)$

$\Phi_{FD} = -T \sum_\ell \log\left(1 + e^{-\beta(E_0 - \mu)}\right)$

Classical (MB) physics should emerge from quantum statistics when # of accessible ~~evel~~ energy levels $\gg N$

Recall canonical spin system $\rightarrow$ p_i for $E_i > E_0$ exponentially suppressed for low T

$$p_i = \frac{1}{Z} e^{-E_i/T} \rightarrow 0$$

$\rightarrow$ Need <u>high T</u> for classical emergence

Let's compute $\langle N \rangle = \dfrac{-\partial \Phi}{\partial \mu}$ for all three & check high T

$$\boxed{\begin{aligned}
\langle N \rangle_{MB} &= T \sum_l \frac{\partial}{\partial \mu} e^{-\beta(E_l - \mu)} = \sum_l \frac{1}{e^{\beta(E_l-\mu)}} = \sum_l \langle n_l \rangle_{MB} \\
\langle N \rangle_{BE} &= -T \sum_l \frac{-\beta e^{-\beta(E_l-\mu)}}{1 - e^{-\beta(E_l-\mu)}} = \sum_l \frac{1}{e^{\beta(E_l-\mu)} - 1} = \sum_l \langle n_l \rangle_{BE} \\
\langle N \rangle_{FD} &= T \sum_l \frac{\beta e^{-\beta(E_l-\mu)}}{1 + e^{-\beta(E_l-\mu)}} = \sum_l \frac{1}{e^{\beta(E_l-\mu)} + 1} = \sum_l \langle n_l \rangle_{FD}
\end{aligned}}$$

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- $\langle n_l \rangle_{MB} = \dfrac{1}{e^{\beta(E_l-\mu)}}$
- $\langle n_l \rangle_{BE} = \dfrac{1}{e^{\beta(E_l-\mu)} - 1}$
- $\langle n_l \rangle_{FD} = \dfrac{1}{e^{\beta(E_l-\mu)} + 1} \leq 1$ ✓

All occupation #s ≥ 0 ✓

All agree when $e^{\beta(E_l-\mu)} \gg 1 \rightarrow \langle n_l \rangle \ll 1$ ✓

Classical limit $\beta(E_l - \mu) \gg 1$ high T???

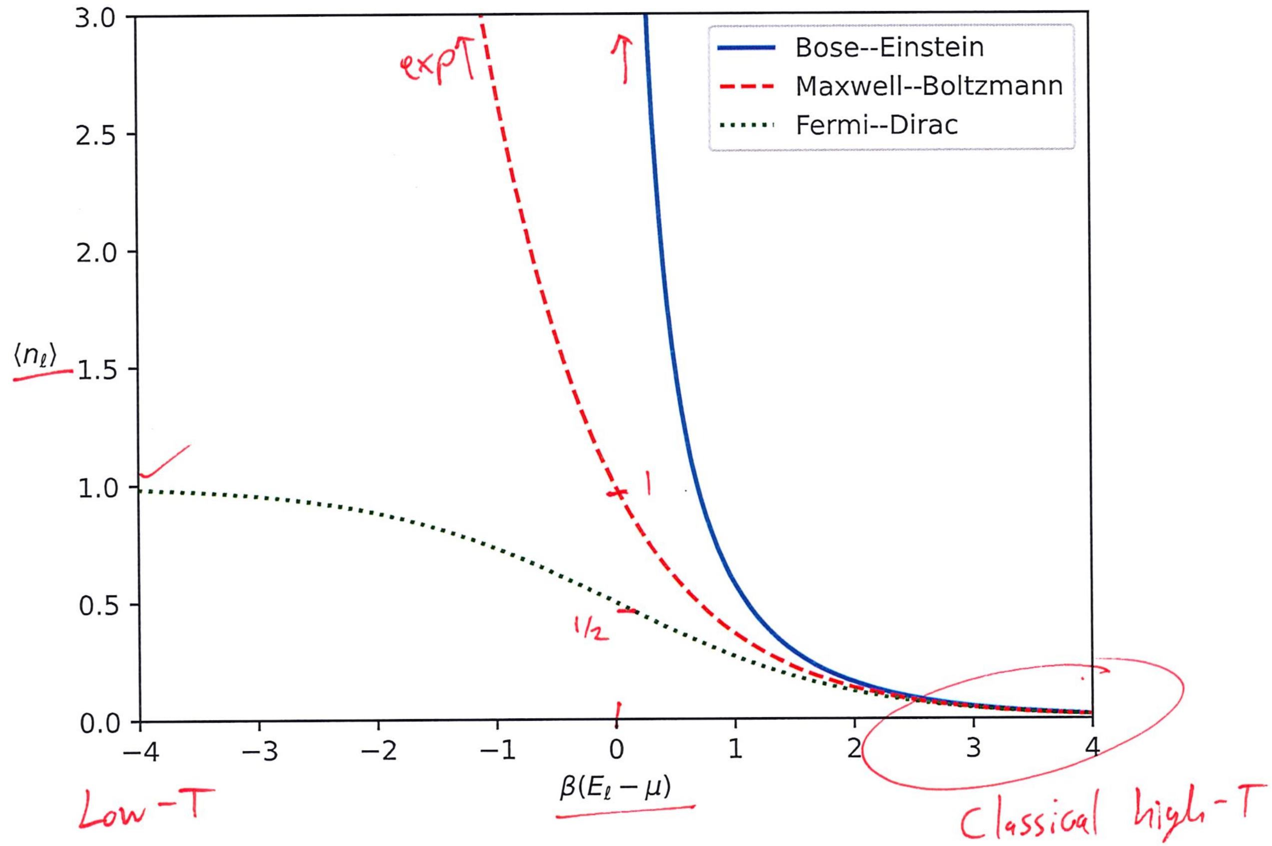
<u>$\beta \to 0$</u> w/ fix <u>$(E_l - \mu)$</u>

$\langle n_l \rangle_{MB} \approx 1$ $\quad \langle n_l \rangle_{FD} \approx \frac{1}{2}$ $\quad \langle n_l \rangle_{BE} \rightarrow \infty$

True classical limit is $\{T, -\mu\} \rightarrow \infty$ s.t. $-\mu \gg T \gg E_l$

fewer particles

more accessible energy levels



Bose--Einstein
Maxwell--Boltzmann
Fermi--Dirac
$\langle n_\ell \rangle$
$\beta(E_\ell - \mu)$
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−4
−3
−2
−1
0
1
2
3
4
exp
1
1/2
Low-T
Classical high-T

Look ahead to quantum gases — $\begin{cases} \text{grand-canonical ensemble} \\ \text{quantum statistics} \end{cases}$

Need energies E_ℓ going into $\mathcal{Z}_g^{BE}$ and $\mathcal{Z}_g^{FD}$

Two limiting cases

non-relativistic

$$E = \frac{p^2}{2m}$$

$$= \frac{\hbar^2 \pi^2}{2mL^2} (k_x^2 + k_y^2 + k_z^2)$$

$$k_{x,y,z} = 1, 2, 3, \ldots$$

Heisenberg uncertainty

$$(\Delta x)(\Delta p_x) \gtrsim \hbar$$

$$\underbrace{\Delta x}_{\le L}$$

$$\Delta p_x \gtrsim \frac{\hbar}{L}$$

ultra-rel.

$$m \to 0$$

$$E = c\underbrace{p}_{\sqrt{p_x^2 + p_y^2 + p_z^2}} = \hbar c \frac{\pi}{L} k$$

Waves → k is wavenumber

$\frac{2L}{k} = \lambda$ is quantized wavelength

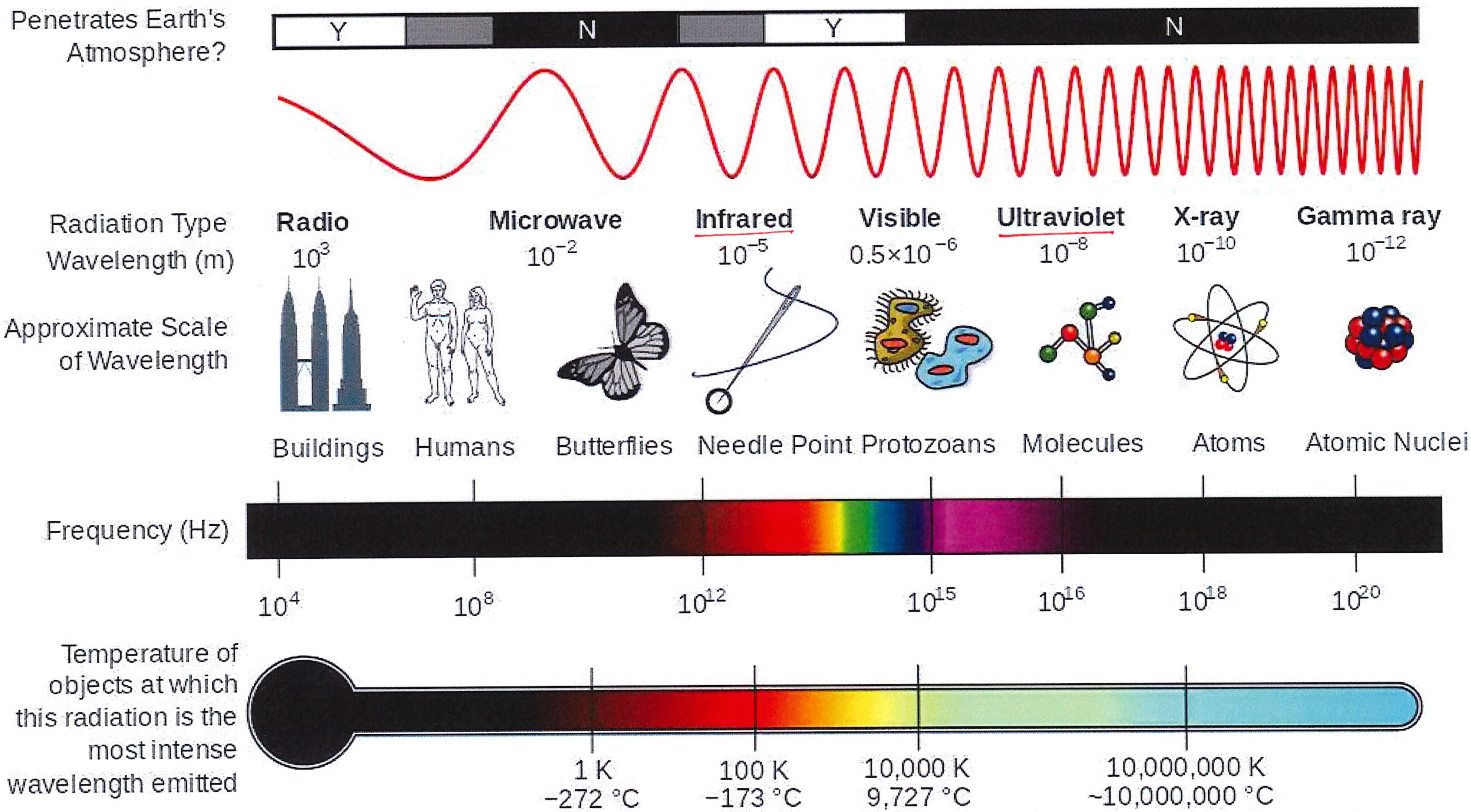
$$\omega = 2\pi f$$

$$= \frac{2\pi}{\lambda} c = c \frac{\pi}{L} k$$

$$E = \hbar \omega$$

Large E → ~~low~~ high freq
short λ
"ultraviolet"

Small E → low freq.
long λ
"infrared"



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