20 Feb supplement

Micro-canonical ensemble temperature

$$\frac{1}{T} = \frac{\partial S}{\partial E}\bigg|_N = \frac{\partial}{\partial E} \log M \bigg|_N$$

Example: Spin system

$$E = -H(2n_+ - N)$$

H $\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow$

$n_+ = 4 \quad n_- = 2$

Different (conserved) E

↳ different n_+, M, S, T

Lowest energy: H $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$

$n_+ = N \quad n_- = 0$

$E_0 = -NH \quad M(E_0) = 1$

Next-lowest energy:

H $\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow$

$\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow$

$n_+ = N-1 \quad n_- = 1$

$E_1 = -(N-1)H + H$

$= -(N-2)H$

page 35

$$M(E_1) = 6 = N = \binom{N}{1}$$

Generalizing, $M(E_{n_-}) = \binom{N}{n_-} = \binom{N}{n_+} = \dfrac{N!}{n_+!(N-n_+)!}$

For temperature, need derivative of $S = \log M$

Need n_+ in terms of (E, N)

Need differentiable approx. to Factorial

One option: ~~log(M) ≈ Nlo~~

$\log(N!) \approx N \log N - N$

Model spin system as a random walk in energy space

Each spin is a "step" of ± 1 in $x = \frac{-E}{H} = 2n_+ - N$

Reduces to simple 1d example, with $p = q = \frac{1}{2}$

All 2^N "walks" (spin config) equally likely

$$M(E_{n_-}) = 2^N P_{n_+} \qquad P_{n_+} = \binom{N}{n_+} \Big/ 2^N$$

Determine $\underline{P_{n_+}}$ from central limit theorem $N \gg 1$

$$p(x) \approx \frac{1}{\sqrt{2\pi N \sigma^2}} \exp\left[-\frac{(x - N\mu)^2}{2N\sigma^2}\right]$$

$$\mu = 2p - 1 = 0$$

$$\sigma^2 = 4pq = 1$$

$$p(x) \approx \frac{1}{\sqrt{2\pi N}} \exp\left(\frac{-x^2}{2N}\right)$$

Integrate distribution using constant approximation

$$2n_+ - N = \frac{-E}{H}$$

$$P_{const}(n_+) = p(2n_+ - N)\underbrace{\Delta n_+}_{1}$$

$$= \frac{1}{\sqrt{2\pi N}} \exp\left(\frac{-E^2}{2NH^2}\right)$$

So $M(E) \approx \dfrac{2^N}{\sqrt{2\pi N}} \exp\left(\dfrac{-E^2}{2NH^2}\right)$

$$\boxed{\rightarrow \frac{1}{T} = \frac{\partial}{\partial E} \log M = \frac{\partial}{\partial E}\left(\frac{-E^2}{2NH^2} + E\text{-indep.}\right) = \frac{-E}{NH^2}}$$

page 36

The temperature is $T \approx \dfrac{-NH^2}{E}$ for $N \gg 1$

(ill-defined if $H = 0 \rightarrow E = 0$)

Diverges if $E \to 0$ $(n_+ \approx n_-)$

Negative if $E > 0$ $(n_+ < n_-)$

↳ Adding energy reduces # of micro-states

Natural systems have $T > 0$ $(E < 0,\ n_+ > n_-)$

Minimum temperature $T_{min} = -H$ for $E_0 = -NH$

Adding energy increases temperature