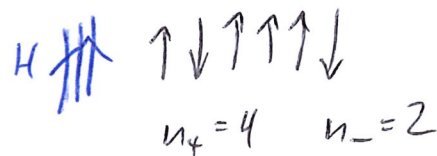


Micro-canonical ensemble temperature

$$\frac{1}{T} = \left. \frac{\partial S}{\partial E} \right|_N = \left. \frac{\partial}{\partial E} \log M \right|_N$$

Example: Spin system

$$E = -H(2n_+ - N)$$



Different (conserved) E

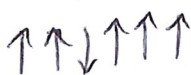
↳ different n_+ , M , S , T

Lowest energy:

$$n_+ = N \quad n_- = 0$$

$$E_0 = -NH \quad M(E_0) = 1$$

Next-lowest energy:



$$n_+ = N-1 \quad n_- = 1$$

$$E_1 = -(N-1)H + H = -(N-2)H$$

$$M(E_1) = 6 = N = \binom{N}{1}$$

Generalizing,

$$M(E_{n_-}) = \binom{N}{n_-} = \binom{N}{n_+} = \frac{N!}{n_+!(N-n_+)!}$$

For temperature, need derivative of $S = \log M$

Need n_+ in terms of (E, M)

Need differentiable approx. to factorial

One option: ~~$\log(M) \approx N \log 2$~~

$$\log(N!) \approx N \log N - N$$

Model spin system as a random walk in energy space

Each spin is a "step" of ± 1 in $x = \frac{-E}{H} = 2n_+ - N$

Reduces to simple 1d example, with $p = q = \frac{1}{2}$

All 2^N "walks" (spin config) equally likely

$$M(E_{n_+}) = 2^N P_{n_+} \quad P_{n_+} = \binom{N}{n_+} / 2^N$$

Determine P_{n_+} from central limit theorem $N \gg 1$

$$p(x) \approx \frac{1}{\sqrt{2\pi N \sigma^2}} \exp\left[-\frac{(x - N\mu)^2}{2N\sigma^2}\right]$$

$$\mu = 2p - 1 = 0$$

$$\sigma^2 = 4pq = 1$$

$$p(x) \approx \frac{1}{\sqrt{2\pi N}} \exp\left(-\frac{x^2}{2N}\right)$$

Integrate distribution using constant approximation

$$P_{\text{const}}(n_+) = p(2n_+ - N) \Delta n_+$$

$$2n_+ - N = \frac{-E}{H}$$

$$= \frac{1}{\sqrt{2\pi N}} \exp\left(\frac{-E^2}{2NH^2}\right)$$

$$S_0 \quad M(E) \approx \frac{2^N}{\sqrt{2\pi N}} \exp\left(\frac{-E^2}{2NH^2}\right)$$

$$\rightarrow \frac{1}{T} = \frac{\partial}{\partial E} \log M = \frac{\partial}{\partial E} \left(\frac{-E^2}{2NH^2} + E\text{-indep.} \right) = \frac{-E}{NH^2}$$

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The temperature is $T \approx \frac{-NH^2}{E}$ for $N \gg 1$

(ill-defined if $H=0 \rightarrow E=0$)

Diverges :if $E \rightarrow 0$ ($n_+ \approx n_-$)

Negative :if $E > 0$ ($n_+ < n_-$)

↳ Adding energy reduces # of micro-states

Natural systems have $T > 0$ ($E < 0$, $n_+ > n_-$)

Minimum temperature $T_{\min} = -H$ for $E_0 = -NH$

Adding energy increases temperature