Thu 2 Feb 143422

Place £5 bet on black

IF ball in black win £10

$$P_{win} = P_{black} = \frac{18}{37}$$

$$P_{lose} = 1 - \frac{18}{37} = \frac{19}{37}$$

$$G_W = 10W - 5N$$

order doesn't matter

$$P_W = \binom{N}{W} P_{win}^{W} P_{lose}^{N-W} = \binom{N}{W} \frac{18^W \, 19^{N-W}}{37^N}$$

$N=5$, $W=0$: $P_0 = \left(\frac{19}{37}\right)^5 \approx 3.6\%$ to lose £25 $(G=-25)$

$W=5$: $P_5 = \left(\frac{18}{37}\right)^5 \approx 2.7\%$ to win £25

Check: All six add up to 100%

CLT: $$p(g) = \frac{1}{\sqrt{2\pi N\sigma^2}} \exp\left[\frac{-(g - N\mu)^2}{2N\sigma^2}\right]$$

$$\mu = \frac{-5}{37} \approx -0.135$$

$$\sigma^2 = \underbrace{25}_{\langle x^2\rangle} - \left(\frac{5}{37}\right)^2 = 25\left(1 - \frac{1}{37^2}\right) \approx 24.98$$

$$\langle gain \rangle = \sum_{x \in A} 5\left(\frac{18}{37}\right) + (-5)\left(\frac{19}{37}\right) = \frac{-5}{37}$$

MATH327: Statistical Physics, Spring 2023
Tutorial activity — Central limit theorem

To illustrate the central limit theorem, let's consider the roulette wheel introduced in class (and on page 13 of the lecture notes). A simple game of roulette would let us place bets on whether or not the ball will end up in a red- or black-coloured pocket: If we bet correctly we get back twice the money we put in; otherwise we lose our money. We'll define our (potentially negative) *gain* to be the amount we receive minus the amount we spend on bets.

- Suppose we place £5 bets on 'black' for each of N spins of the roulette wheel. What are the probabilities and gains of winning and of losing for any single one of those spins? Letting $W = 0, \cdots, N$ be the number of spins where we win, what is our total gain G_W as a function of (N, W)?

- Recall that the number of different ways we could win W times out of N attempts is given by the binomial coefficient

$$\binom{N}{W} = \frac{N!}{W!\,(N-W)!},$$

  with $0! = 1$. Setting $N = 5$, what are the six probabilities p_0 through p_5 that we win $W = 0, \cdots, 5$ times? What is the general p_W for any (N, W)?

- Now let's apply the central limit theorem to this setup. What are the mean gain and its variance for a single spin of the wheel? What is the resulting probability distribution $p(g)$ given by the central limit theorem for the gain $g \in \mathbb{R}$ after $N \gg 1$ spins?

- In order to compare the distribution $p(g)$ against the probabilities p_W considered above, we need to integrate $p(g)$ over appropriate intervals as discussed in class (and on page 16 of the lecture notes). Natural intervals to consider are

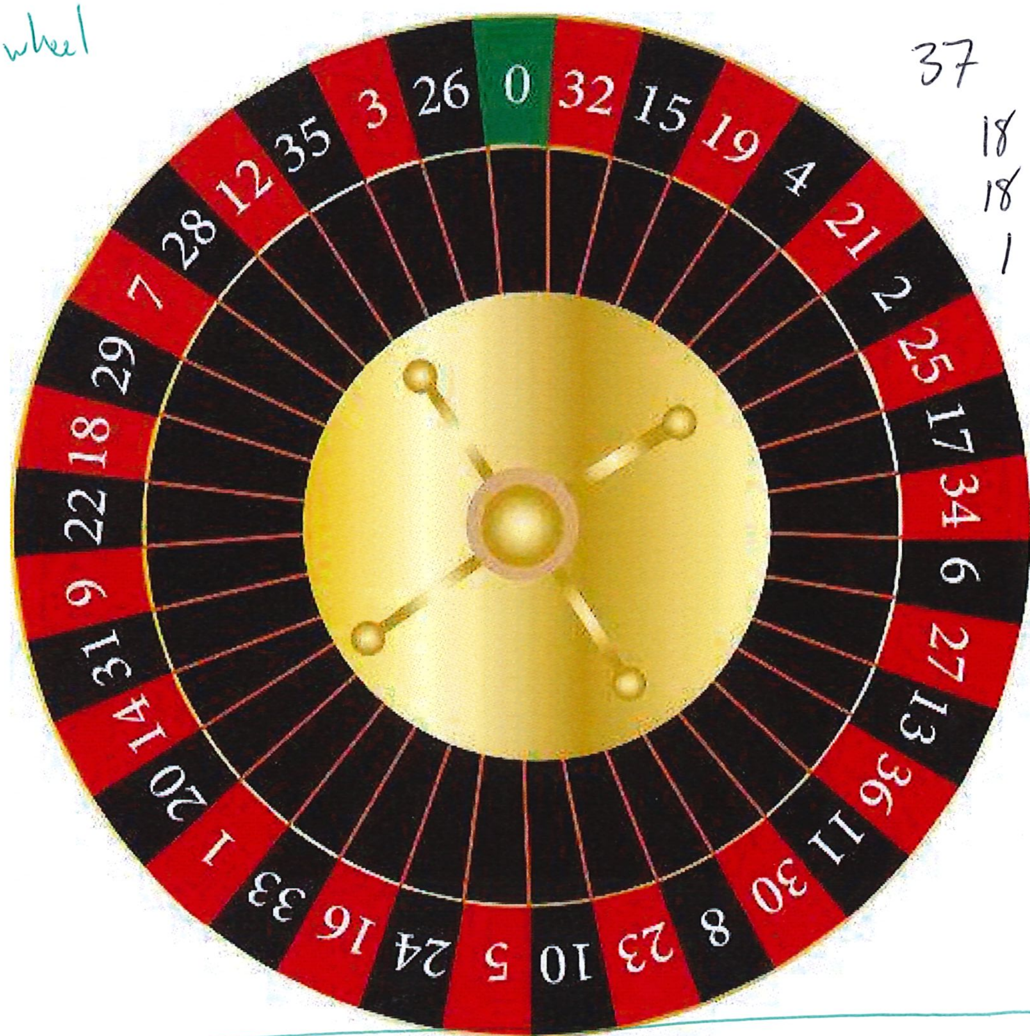
$$P_{\text{integ}}(G_W) \equiv \int_{G_W - \Delta G/2}^{G_W + \Delta G/2} p(g)\, dg,$$

  where $\Delta G = G_{W+1} - G_W$ is a constant you can read off from your work so far. These numerical integrations are not convenient to do by hand, but can easily be performed by Maple, Python, MATLAB, Mathematica, etc. Alternately, we can simplify further by approximating $p(g)$ as a constant within each interval, which would give us

$$P_{\text{const}}(G_W) \equiv p(G_W)\, \Delta G.$$

  Keeping $N = 5$, what are the six P_{integ} and P_{const}?

Roulette wheel



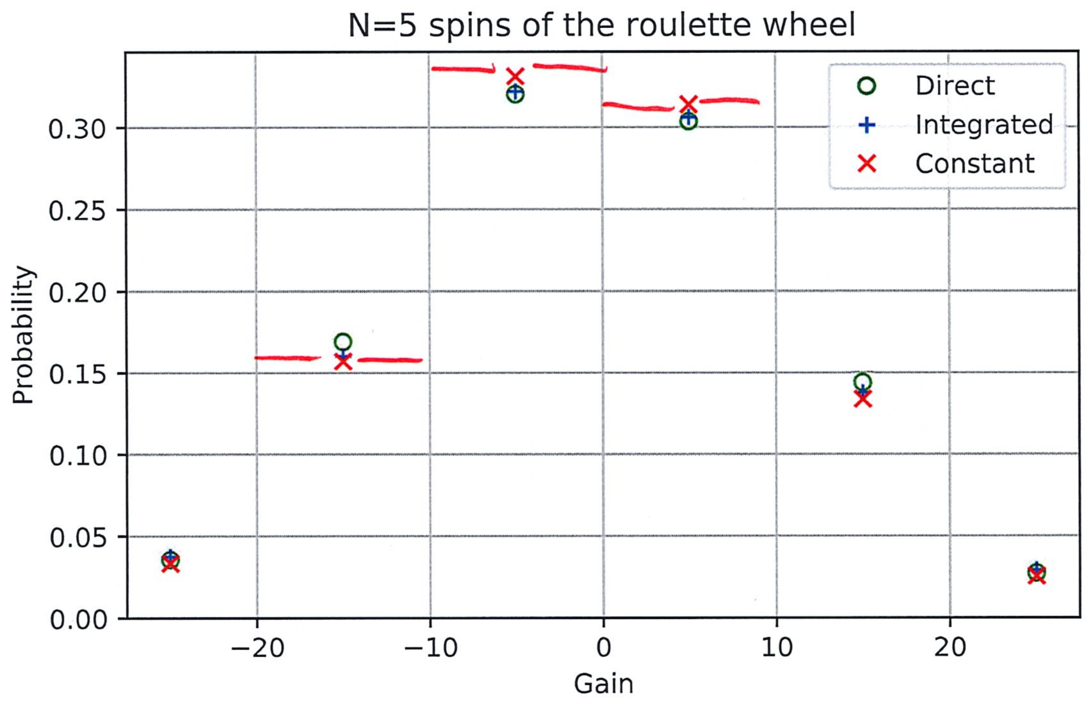
37 pockets
18 red
18 black
1 green

Spin wheel, measure pocket $\rightarrow$ $A = \{0, 1, 2, \dots 36\}$

Fair wheel $\rightarrow$ $\sum_{i=0}^{36} p = 1$ $\quad p = 1/37$ for each pocket

$\mathcal{F} = \{\text{red}, \text{black}, \text{green}\}$ (mutually exclusive)

$p_{red} = \frac{18}{37}$ $\quad p_{black} = \frac{18}{37}$ $\quad p_{green} = 1/37$


N=5 spins of the roulette wheel
Direct
Integrated
Constant
Probability
0.00
0.05
0.10
0.15
0.20
0.25
0.30
−20
−10
0
10
20
Gain