Tuesday, 31 Jan 258503

Recap

Emergent behaviour from many particles

Measurement → mathematical control over observations

Probability spaces $(A, \mathcal{F}, P)$

Mean $\mu = \langle x \rangle = \sum_{X \in A} X \, P(X)$

Variance $\sigma^2 = \langle (X-\mu)^2 \rangle = \sum_{X \in A} (X-\mu)^2 \, P(X)$

Plan

Law of large numbes

Central limit theorem

Start random walks & diffusion

Standard deviation

$\sigma = \sqrt{\sigma^2}$

$$\boxed{\begin{aligned} \sigma^2 = \langle (X-\mu)^2 \rangle &= \langle (X - \langle X \rangle)^2 \rangle \\ &= \langle X^2 - 2X\langle X \rangle + \langle X \rangle^2 \rangle \\ &= \langle X^2 \rangle - 2\langle X \rangle\langle X \rangle + \langle X \rangle^2 \\ &= \langle X^2 \rangle - \langle X \rangle^2 \\ \sigma &= \sqrt{\langle X^2 \rangle - \langle X \rangle^2} \end{aligned}}$$

page 14

Repetition

New experiment repeats $\mathcal{E}$ $\to$ outcome A R time

$\hookrightarrow$ outcome space B

$A = \{X_1, X_2 \dots X_n\}$ $R=4$

$B = \{X_1 X_1 X_1 X_1, \; X_{47} X_{73} X_1 X_n, \; \dots\}$

$\#A = n$

$\#B = \underbrace{n \cdot n \cdots n}_{R \text{ times}} = n^R$

page 14

$X_i X_j X_h \cdots X_\ell \in B$ determined by R $X^{(r)} \in A$

mean $\langle X^{(r)} \rangle = \mu$

variance $\langle (X^{(r)} - \mu)^2 \rangle = \sigma^2$

$$P_B(X_i X_j X_h X_\ell) = P_A(X_i) P_A(X_j) P_A(X_h) P_A(X_\ell)$$

Arithmetic mean $\overline{X}_R = \frac{1}{R} \sum_{r=1}^{R} X^{(r)}$ average over R

random variable of repeated experiment

relate to μ of <u>single experiment</u>

assume finite μ, σ^2

Consider $\left\langle (\overline{X}_R - \mu)^2 \right\rangle = \left\langle \left(\frac{1}{R} \sum_{r=1}^{R} X^{(r)} - \mu \right)^2 \right\rangle$ $\quad R\mu = \sum_r \mu$

$$= \frac{1}{R^2} \left\langle \left(\sum_r (X^{(r)} - \mu) \right)^2 \right\rangle$$

$$= \frac{1}{R^2} \left\langle \left(\sum_r (X^{(r)} - \mu) \right) \left(\sum_s (X^{(s)} - \mu) \right) \right\rangle$$

$$\left(\sum_i a_i \right) \left(\sum_j b_j \right) = \sum_{i,j} a_i b_j$$

$$= \frac{1}{R^2} \left\langle \sum_{r,s} \underbrace{(X^{(r)} - \mu)(X^{(s)} - \mu)}_{\sigma^2 \delta_{rs}} \right\rangle$$

$$= \frac{\sigma^2}{R^2} \sum_{r,s} \delta_{rs} = \frac{\sigma^2}{R^2} \sum_r 1 = \frac{\sigma^2}{R^2} R = \frac{\sigma^2}{R}$$

page 15

$$\xrightarrow[R \to \infty]{} 0$$

Vanishing sum of squares → every term zero

$$\lim_{R \to \infty} \left(\frac{1}{R} \sum_{r=1}^{R} X^{(r)} - \mu \right) = 0$$

$$\lim_{R \to \infty} \overline{X}_R = \mu$$

large of lange numbers

Probability distribution

Continuous X, uncountable A

discrete sums → continuous integrals

Probability density must be integrated to give probability

$$P(a \leq X \leq b) = \int_a^b p(x)\, dx \qquad x \in \mathbb{R}$$

Expectation values

$$\langle f(x) \rangle = \int f(x)\, p(x)\, dx$$

(whole range implicitly)

$$\langle 1 \rangle = \int p(x)\, dx = 1$$

$$\langle x^l \rangle = \int x^l\, p(x)\, dx$$

$l = 1 \rightarrow$ mean $\mu = \langle x \rangle$

variance $\sigma^2 = \underbrace{\langle x^2 \rangle}_{l=2} - \mu^2$

Central limit theorem

Consider N random variables $x_1, x_2, \ldots x_N$

Identically distributed — same finite μ, σ^2 for each x_i

(repeating experiment, for example)

Like arithmetic mean, sum is also random variable

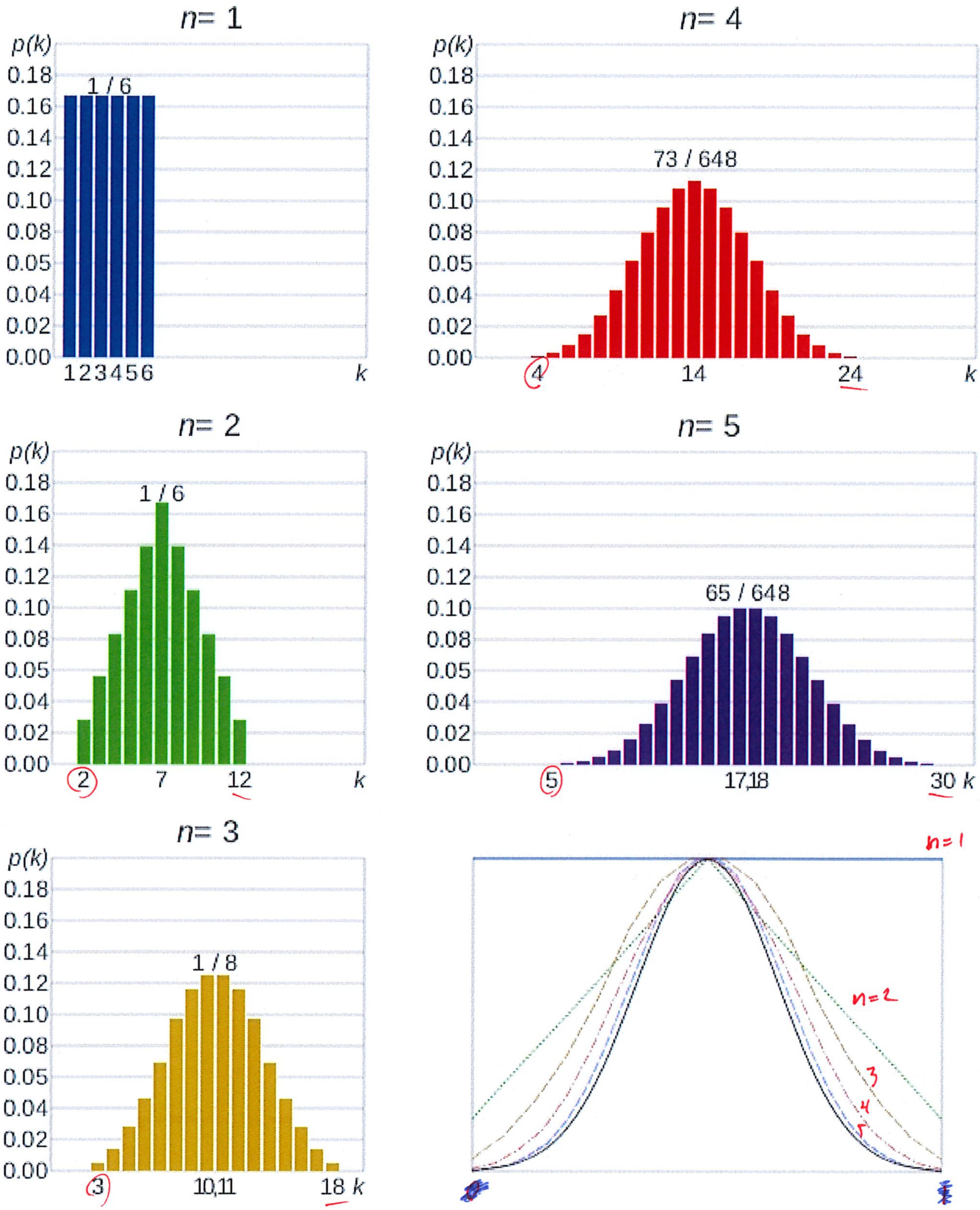
$$s = \sum_{i=1}^{N} x_i$$

gaussian

For $N \gg 1$ prob. distribution $p(s) \approx \frac{1}{\sqrt{2\pi N \sigma^2}} \exp\left[-\frac{(s - N\mu)^2}{2N\sigma^2} \right]$

Collective behaviour from μ, σ^2

$$\int p(s)\, ds = 1$$



commons.wikimedia.org/wiki/File:Dice_sum_central_limit_theorem.svg