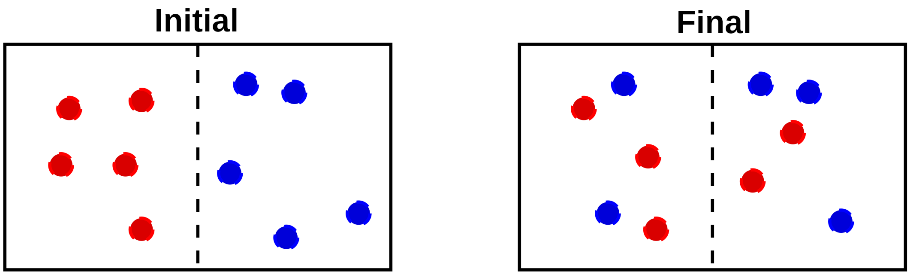
MATH327: Statistical Physics, Spring 2022

Tutorial problem — Mixing entropy

Let's consider a slight variation to the particle exchange thought experiment we worked through in class. We again begin with two canonical ideal gases, initially separated by a wall, each with N particles in volume V at temperature T. **All** $2N$ particles have identical physical properties, *except* that those initially in the left compartment (the "reds") are distinguishable from those in right compartment (the "blues") by their colour. Call this initial system Ω_0. We have already computed its entropy $S_0 = 2S_I(N, V) = 5N + 2N \log\left(\frac{V}{N\lambda_{\text{th}}^3}\right)$, where $\lambda_{\text{th}} = \sqrt{2\pi\hbar^2/(mT)}$.

We then carry out the procedure of removing the wall, allowing the combined system to reach thermodynamic equilibrium, and then re-inserting the wall to re-separate the two systems. Call the combined system Ω_C with entropy S_C. As discussed in class, it's safe to assume that N particles end up in each of the two re-separated systems. However, red and blue particles can now appear in either of the two re-separated systems. Call this final system Ω_F with entropy S_F. The initial and final systems are illustrated by the figure below.



The first task is to compute the mixing entropy $S_{\text{mix}} = S_C - S_0$, where in the combined system Ω_C we now have two sets of N indistinguishable particles, but can distinguish between the two sets. The starting point is the partition function

$$Z_C = \frac{1}{N!}\frac{1}{N!}Z_1^{2N} = \frac{1}{N!}\frac{1}{N!}\left(\frac{2V}{\lambda_{\text{th}}^3}\right)^{2N},$$

where $Z_1 = 2V/\lambda_{\text{th}}^3$ is the single-particle partition function.

The second task is to compute the final entropy S_F, to see whether $S_F \geq S_C$ as demanded by the second law of thermodynamics. We can break this up into two steps. The first of these is to compute the partition function Z_F of the two re-separated systems (each with N particles), summing over all ways of dividing the red and blue particles between them. The following special case of the Zhu–Vandermonde identity may be useful for this step:

$$\sum_{k=0}^{N} \binom{N}{k}^2 = \binom{2N}{N}.$$

Finally, use your result for Z_F to determine the final entropy S_F.