MATH327: Statistical Physics, Spring 2022

Tutorial problem — Entropy bounds

We met the second law of thermodynamics by considering what happens when two spin (sub)systems are brought into thermal contact with each other — allowed to exchange energy but not particles. Conservation of energy means that if subsystem Ω_1 has energy e_1, the other subsystem Ω_2 must have energy $E - e_1$, where E is the total energy of the micro-canonical combined system Ω. We found (in Eq. 21 on page 34 of the lecture notes) that the total number of micro-states of the overall system is

$$M = \sum_{e_1} M_{e_1}^{(1)} M_{E-e_1}^{(2)}$$

where $M_e^{(S)}$ is the number of micro-states of subsystem $S \in \{1, 2\}$ with energy e.

Because M is a sum of strictly positive terms, we can easily set bounds on it. Say the sum over e_1 has $N_{\text{terms}} \geq 1$ terms $M_{e_1}^{(1)} M_{E-e_1}^{(2)}$, and define $\max$ be the largest of those terms. Then $\max \leq M$, with equality holding when there is only one term in the sum. Similarly, $M \leq N_{\text{terms}} \cdot \max$, with equality holding when every term in the sum is the same. All together, we have

$$\max \leq M \leq N_{\text{terms}} \cdot \max .$$

This can be more powerful than it may initially appear, thanks to the large numbers involved in statistical physics. For illustration, suppose $\max \sim e^N$ and $N_{\text{terms}} \sim N$ for a system with N degrees of freedom. (We have already seen $M = 2^N = e^{N \log 2}$ for a system of N spins with $H = 0$, while $H > 0$ introduces factors of $N!$ that Stirling's formula can recast in terms of $N^N = e^{N \log N}$.) Then we have $e^N \lesssim M \lesssim N e^N$. If we take the logarithm and recall $\log M = S$ is the entropy, this becomes $N \lesssim S \lesssim N + \log N$. With our characteristic $N \sim 10^{23}$, we have $\log N \sim 50$ and $10^{23} \lesssim S \lesssim 10^{23} + 50$, a very tight range in relative terms, with the upper bound only $\sim 10^{-20}\%$ larger than the lower bound.

To see how this works in practice, let each of Ω_1 and Ω_2 be a spin system with $N_1 = N_2 = 10$ spins and $H = 1$. Fix $E = -10$ for the combined system and numerically compute the bounds on its entropy,

$$\log(\max) \leq S \leq \log(N_{\text{terms}} \cdot \max) .$$

What fraction of the true entropy S is accounted for by $\log(\max)$? How do these answers change for $N_1 = N_2 = 20$, 30, 40, $\cdots$, still with fixed $E = -10$?