MATH327: Statistical Physics, Spring 2022

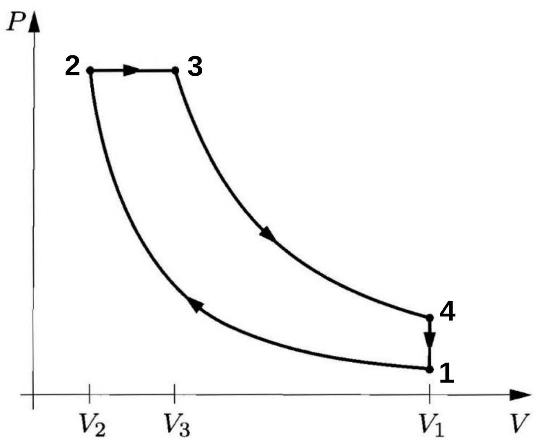
Homework assignment 2

Instructions

Complete all four questions below and submit your solutions by file upload on Canvas.[1] Clear and neat presentations of your workings and the logic behind them will contribute to your mark. This assignment is **due by 23:59 on Thursday, 5 May**, and anonymous marking is turned on.

Question 1: Thermodynamic cycle

Consider the Diesel cycle defined by the PV diagram shown below, in which the 'compression' stage $1 \to 2$ and the 'power' stage $3 \to 4$ are both adiabatic, while the pressure is constant during the 'injection/ignition' stage $2 \to 3$.



Calculate the efficiency of the Diesel cycle, η_D, in terms of the compression ratio $r \equiv V_1/V_2 > 1$ and the cutoff ratio $C \equiv V_3/V_2 > 1$, where $C < r$.

[10 marks]

Fixing the compression ratio r, compare η_D to the efficiency of the Otto cycle. Is the Diesel cycle more efficient than the Otto cycle, less efficient, or the same? How does this depend on the cutoff ratio C?

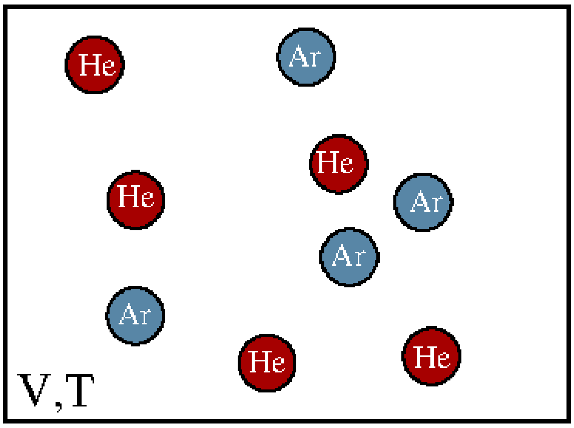
[4 marks]

[1] By submitting solutions to this assessment you affirm that you have read and understood the Academic Integrity Policy detailed in Appendix L of the Code of Practice on Assessment and have successfully passed the Academic Integrity Tutorial and Quiz. The marks achieved on this assessment remain provisional until they are ratified by the Board of Examiners in June 2022.

Question 2: Mixed ideal gases

Consider a mixture of two ideal (non-interacting) gases in thermodynamic equilibrium in a container of volume V at temperature T, like that illustrated below. Let N_1 and N_2 be the fixed particle numbers of the two gases. Within each gas the particles are indistinguishable, but particles of one gas are distinguishable from particles of the other gas. In particular, they have different masses m_1 and m_2, implying different thermal de Broglie wavelengths and single-particle canonical partition functions:

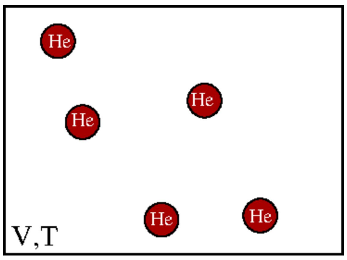
$$\lambda_i(T) = \sqrt{\frac{2\pi\hbar^2}{m_i T}} \qquad\qquad Z_1^{(i)}(T) = \frac{V}{\lambda_i^3}.$$

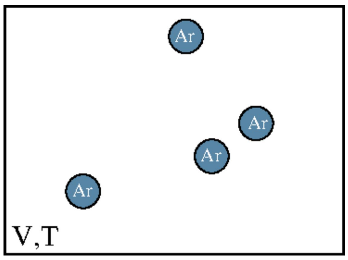


(a) Calculate the canonical partition function Z and the Helmholtz free energy of the $(N_1 + N_2)$-particle mixture, approximating $\log(N_i!) \approx N_i \log N_i - N_i$.
[4 marks]

(b) Calculate the internal energy $\langle E \rangle$ and the entropy S of the mixture. What is the condition of constant entropy?
[4 marks]

(c) Calculate the pressure P of the mixture, and relate it to the pressures P_1 and P_2 of each gas in isolation (as illustrated below).
[4 marks]





Question 3: Particle number fluctuations

Consider the fugacity expansion of the grand-canonical partition function (Eq. 82),

$$Z_g(T, \mu) = \sum_{N=0}^{\infty} \xi^N \, Z_N(T),$$

where the fugacity $\xi = e^{\beta\mu} = e^{\mu/T}$ and $Z_N(T)$ is the N-particle canonical partition function (which is independent of ξ). Recall that $\Phi(T, \mu) = -T \log Z_g(T, \mu)$ is the corresponding grand-canonical potential.

(a) Derive a relation between the average particle number $\langle N \rangle$ and the derivative $\frac{\partial}{\partial \log \xi}\Phi = \xi \frac{\partial}{\partial \xi}\Phi$.

[4 marks]

(b) Derive a relation between $\left\langle (N - \langle N \rangle)^2 \right\rangle$ and $\left(\xi \frac{\partial}{\partial \xi}\right)^2 \Phi$.

[4 marks]

(c) Specializing to Maxwell–Boltzmann statistics, for which the fugacity expansion simplifies to $Z_g^{\text{MB}}(T, \mu) = \exp[\xi Z_1(T)]$, show

$$\frac{\sqrt{\left\langle (N - \langle N \rangle)^2 \right\rangle}}{\langle N \rangle} = \frac{1}{\sqrt{\langle N \rangle}}.$$

[4 marks]

As an aside, this final result means that the relative fluctuations in the particle number vanish in the **thermodynamic limit** $\langle N \rangle \to \infty$. That is, when $\langle N \rangle$ is large it is approximately constant, which allows the grand-canonical system to be approximated by the corresponding canonical system with fixed N.

Question 4: Magnetization

Consider a system of N (distinguishable) non-interacting 'spins' in a lattice at temperature T, where the value s_i of each spin can vary *continuously* in the range $-1 \leq s_i \leq 1$. In an external magnetic field of strength $H > 0$, the internal energy of the system is $E = -H \sum_{i=1}^{N} s_i$.

(a) Calculate the canonical partition function Z and the Helmholtz free energy of the system.

[4 marks]

(b) Calculate the magnetization $\langle m \rangle$ of the system. For finite $H > 0$, what are its low- and high-temperature limits, $\lim_{T \to 0} \langle m \rangle$ and $\lim_{T \to \infty} \langle m \rangle$?

[4 marks]

(c) Calculate the leading T-dependent correction to each of the low- and high-temperature limits from the previous part.

[4 marks]