Statistical Physics 2019/20
MATH327
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Ising model addenda

Friday 1 May

Plan

Today: Lecture loose ends, tutorial on sample exam

Next Tuesday, 5 May: Module review for exam revision

Monday, 11 May: Optional exam revision

Friday, 29 May: Exam available 9AM, due in 24 hours

Questions?

Recap

Mean-field approximation assumes small fluctuations on average
$\longrightarrow$ turn Ising model into modified non-interacting system

Produces **self-consistency condition** for magnetization order parameter

Predicts second-order transition at critical $\beta_c = \dfrac{1}{2d}$
where $\langle m \rangle \propto (T_c - T)^{1/2}$ with critical exponent $1/2$

Accuracy of approximation improves as dimension d increases

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- Fails badly compared to exact $d = 1$ solution
- Qualitatively but not quantitatively correct compared to exact $d = 2$ solution
- Exactly reproduces Ising model in formal limit $d \to \infty$
- Numerical methods required to analyze $d \geq 3$ Ising model

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Questions?

Monte Carlo integration / summation (D. V. Schroeder section 8.2)

Simple idea: Randomly select ("sample") points in integration domain
or terms in sum

Add up integrand / summand from these sample points,
normalizing by the coverage of the samples

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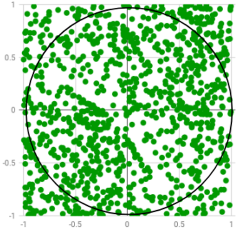
Example: Let's compute

$$\int_{-1}^{1} dx \int_{-1}^{1} dy\, \Theta\left(1 - \left\{x^2 + y^2\right\}\right) = \pi$$



$\longrightarrow$ normalize samples in disk vs. full 2×2 domain

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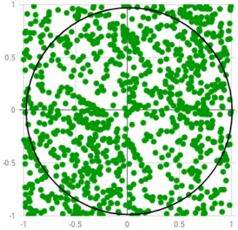

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Monte Carlo is most useful for integration over *many* variables

Monte Carlo importance sampling (D. V. Schroeder section 8.2)

We want to analyze
$$Z(\beta, N, H) = \sum_{\{s_i\}} \exp\left[\beta \sum_{(ij)} s_i s_j + \beta \cdot H \sum_i s_i\right]$$

Even for modest $N = 10 \times 10 \times 10 \times 10 = 10,000$ computers can consider only vanishingly small fraction of $2^N \sim 10^{3000}$ possible configurations $\{s_i\}$

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Solution: Design algorithms to randomly sample configurations i with probability proportional to their **importance** $\sim e^{-\beta E_i}$

Monte Carlo importance sampling (D. V. Schroeder section 8.2)

Metropolis–Rosenbluth–Teller algorithm (1953):

Start with any micro-state i and choose one spin at random

Compute how energy would change if this spin were flipped (ΔE)

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Metropolis–Rosenbluth–Teller algorithm (1953):

Start with any micro-state i and choose one spin at random

Compute how energy would change if this spin were flipped (ΔE)

Flip spin with probability $p = \min\left\{1, e^{-\beta\Delta E}\right\}$ otherwise leave unchanged
$\longrightarrow$ new micro-state $i+1$ (possibly same as before)

Repeat as many times as possible

Example of **Markov process**

Monte Carlo importance sampling (D. V. Schroeder section 8.2)

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Flip spin with probability $p = \min\left\{1, e^{-\beta\Delta E}\right\}$ otherwise leave unchanged

Repeat as many times as possible

Check: Consider micro-states A and B with $E_A \leq E_B$

Relative probabilities of moving between these two micro-states:

$$\frac{\mathcal{P}(A \to B)}{\mathcal{P}(B \to A)} = \frac{\min\left\{1, e^{-\beta(E_B - E_A)}\right\}}{\min\left\{1, e^{-\beta(E_A - E_B)}\right\}} = \frac{e^{-\beta(E_B - E_A)}}{1} = \frac{e^{-\beta E_B}}{e^{-\beta E_A}}$$

proportional to **importance** $\sim e^{-\beta E_i}$ as desired ✓

Quick glimpse: Universality (D. Tong chapter 5)

Recall $\langle m \rangle \propto (T_c - T)^b$ with **critical exponent** b as $T \to T_c^-$

For $d \geq 4$ numerical calculations give $b = 0.5 \longrightarrow$ mean-field result

[cf. arXiv:1202.3031]

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[cf. arXiv:1202.3031]

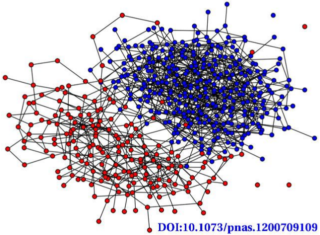
For $d = 3$ numerical calculations give $b = 0.32630(22)$ [cf. arXiv:1806.03558]

This same critical exponent $b \approx 0.326$ characterizes liquid–gas phase transitions and many others

Example of **universality** — essential features independent of specific underlying details

Quick glimpse: Voter models

Interacting spin systems
as basis for modelling social-scientific phenomena


DOI:10.1073/pnas.1200709109

Interpret ‘spins’ as opinions
with ‘energy’ from (dis)agreeing with neighbours

Galam, “Sociophysics” (2012)
Balankin et al., “Ising percolation in a three-state majority vote model” (2016)

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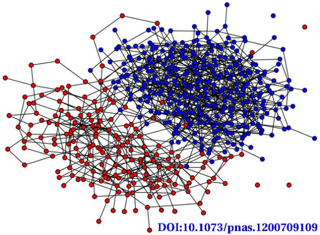

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Many possible generalizations

Individuals interact with (social) network

rather than just nearest neighbours on a lattice

Extended ‘spins’ $s_i \in \{\cdots, -2, -1, 0, +1, +2, \cdots\}$

to model level of committment to opinion

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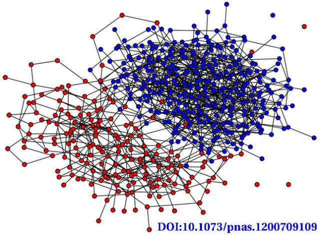


Sample results (with many caveats)

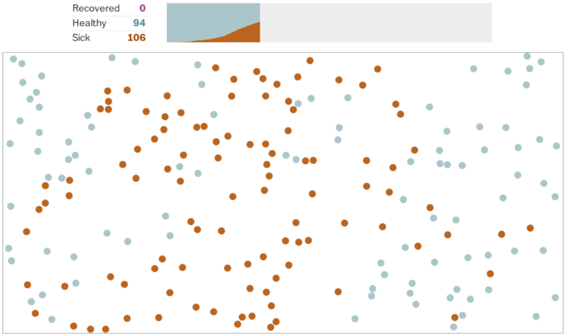
‘Stable non-consensus states’ possible
with ‘clusters’ of aligned voters mostly interacting with each other

Phase transitions possible if opinion reaches ‘critical concentration’

Quick glimpse: Epidemic modeling

Interacting gas as basis for modelling outbreaks and mitigation measures

Interactions potentially spread infection between individuals



washingtonpost.com/graphics/2020/world/corona-simulator

Wrap up

Interacting statistical systems often require numerical analysis

Importance sampling Monte Carlo methods use probabilistic random sampling to approximately evaluate integrals

Essential features of phase transitions exhibit **universality**, don't depend on specific underlying details

Vast ranges of applications for interacting statistical systems across sciences

Questions?