Statistical Physics 2019/20
MATH327
Kurt Langfeld & David Schaich

Anomalous Diffusion & Computer Project Part B

Friday 27 March

Assessment logistics

Alternative final assessment details to be announced via VITAL at 16:00 today

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Week6 homework marks & feedback uploaded to VITAL

$\longrightarrow$ Let me know if anything appears problematic

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Second part of computer project updated for submission via VITAL
$\longrightarrow$ Posted under “Assessment”, due date Friday, 1 May (week 11)

Questions?

HW7 generic feedback

1) Taking derivative $\langle E \rangle = -T^2 \frac{\partial}{\partial T} \left(\frac{F_{\text{indis.}}(T)}{T} \right)$ reasonably straightforward

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2) Most common problem in low-temperature limit $e^{-2\beta H} \ll 1$

was computing only the constant $\lim_{T \to 0} \langle E \rangle = -NH$

HW7 generic feedback

3) Most common problem in high-temperature limit $\beta H \ll 1$

was expanding $$\frac{1}{1-e^{-x}} = \frac{1}{x + \mathcal{O}(x^2)}$$

rather than $$\frac{1}{1-e^{-x}} = \frac{1}{x} + \frac{1}{2} + \frac{x}{12} + \mathcal{O}\left(x^2\right) .$$

The answer was given in Eq. (41) on page 63:

$$\frac{\langle E \rangle}{H} = -\frac{N(N+2)\beta H}{3} + \mathcal{O}\left(\beta^3 H^3\right) .$$

Questions?

HW9 generic feedback

a) Opportunity to think about

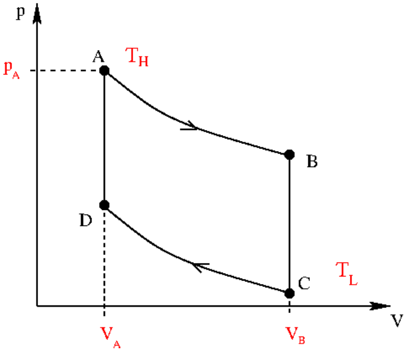
when work done **by** system (*A–B*)

when work done **on** system (*C–D*)

when heat enters system (*A–B* & *D–A*)

& when heat leaves system (*B–C* & *C–D*)

These caused the main issues in parts **c–d**



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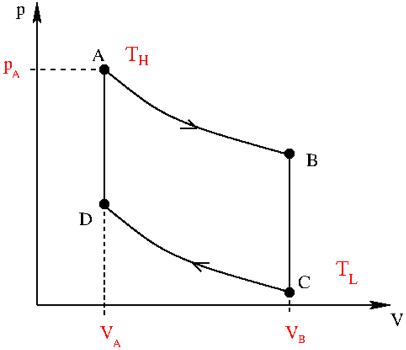


Physics-focused comments:

Isothermal processes are slow, not usable in practical engines

Stirling cycle has same *pV*-diagram plus regenerator

to store heat internally rather than transferring to/from reservoirs

HW9 generic feedback

e) Good checks: Efficiency must always be $0 \leq \eta \leq 1$, with $\eta = 0$ when $T_L = T_H$

Stirling cycle with regenerator has maximal efficiency $\eta = 1 - \dfrac{T_L}{T_H} = \eta_{\text{Carnot}}$

Without regenerator efficiency is lower

Questions?

Anomalous diffusion (page 113)

Still considering "random walker" with position x_N after N steps

(at time $t_N = \Delta t \cdot N$)

New ingredient:

Random step lengths drawn from distribution with **no standard deviation**

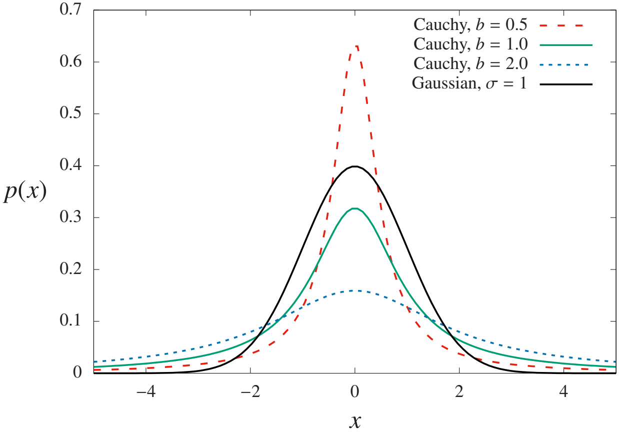
$\longrightarrow$ Central Limit Theorem inapplicable

Cauchy–Lorentz distribution (page 113)

$$p_L(x) = \left(\frac{1}{\pi b}\right) \frac{1}{1 + (x/b)^2}, \qquad \int_{-\infty}^{\infty} p_L(x)\, dx = 1$$

Constant parameter b
controls width of distribution

"Fat tails" at large x



Moments of Cauchy–Lorentz distribution (page 113)

$$\text{Recall} \quad \langle x^N \rangle = \int_{-\infty}^{\infty} x^N \, p_L(x) \, dx = \int_{-\infty}^{\infty} x^N \left(\frac{1}{\pi b} \right) \frac{1}{1 + (x/b)^2}$$

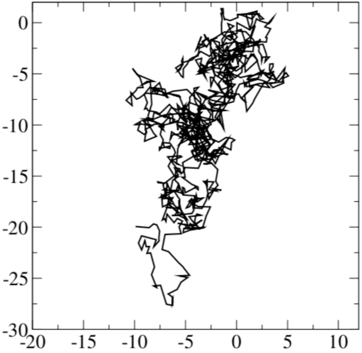
$p_L(x)$ symmetric around zero $\longrightarrow$ $\langle x \rangle = 0$

$\longrightarrow$ Single-step standard deviation would be $\sigma = \langle x^2 \rangle^{1/2}$, but

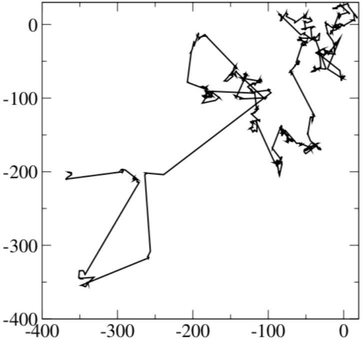
$$\langle x^2 \rangle \propto \int_{-\infty}^{\infty} \frac{x^2}{1 + (x/b)^2} \propto \left[x - b \arctan\left(\frac{x}{b} \right) \right]_{-\infty}^{\infty} \longrightarrow \text{divergent}$$

Ordinary vs. anomalous random walks (pages 113–114)

Gaussian step lengths



Cauchy step lengths



COMMENTS: Ordinary random walk still has $\ell_2 \propto \sqrt{t_N}$ in two dimensions

Anomalous diffusion produces much longer walk

Fat tails of Cauchy distribution $\longrightarrow$ occasional big jumps

Generalized diffusion length (page 114)

Anomalous diffusion still has intrinsic length scale,

$$\ell_\theta = \left\langle |x|^\theta \right\rangle^{1/\theta} \qquad \text{(depending on parameter } \theta\text{)}$$

For Cauchy–Lorentz distribution $0 < \theta < 1$

(absolute value $\longrightarrow \langle |x| \rangle$ for $\theta = 1$ is logarithmically divergent)

Generalized diffusion (page 115)

Recall law of diffusion: $\ell_2(N) = \sqrt{\langle x_N^2 \rangle - \langle x_N \rangle^2} \propto \sqrt{t_N} = t_N^{1/2}$

We obtain this behaviour whenever the standard deviation $\ell_{\theta=2}$ exists

If ℓ_2 does not exist, we can have **super-diffusion** $\ell_\theta \propto t_N^\alpha$ with $\alpha > \frac{1}{2}$

or **sub-diffusion** $\ell_\theta \propto t_N^\alpha$ with $\alpha < \frac{1}{2}$

(not $\alpha < 0 \ldots$)

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Question: Does α depend on θ?

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Question: Does α depend on θ?

Let's find out (empirically) through the computer project...