Young tableaux for SU(3) tensor products 4/12/11

Raising lower indices eliminates symmetry of upper indices

Use Young tableaux to keep track of symmetries

Start with arbitrary tensor (i.e. not necessarily irrep)

$$A^{i_1 \cdots i_n}_{j_1 \cdots j_m} = a^{i_1 \cdots i_n k_1 l_1 k_2 l_2 \cdots k_m l_m} = A^{i_1 \cdots i_n}_{j_1 \cdots j_m} \varepsilon^{j_1 k_1 l_1} \cdots \varepsilon^{j_m k_m l_m}$$

a has $n+2m$ upper indices, but not all the same under interchange

However, applying lowering operators doesn't affect symmetry / antisymmetry (i.e. action of generators treats every (upper) index the same)

For SU(3), the highest weight contained in $A^{i_1 \cdots i_n}_{j_1 \cdots j_m}$ gives ~~are~~ $[n, m]$ irrep generated from $A^{1 \cdots 1}_{2 \cdots 2} \to k_i$ & l_i all 1 or 3

Corresponding Young tableau is symmetric along rows, antisymmetric along columns

Top row contains all 1 indices $(n+m)$, all 3's ${}^{(m)}$ in second row

What about tracelessness? Follows automatically from highest weight procedure

Example: $a^{j_1 j_2 k_1}$ (27 components)

Three-box Young tableaux: $\begin{smallmatrix} j_1 & j_2 \\ k_1 & \end{smallmatrix} \sim a^{j_1 j_2 k_1} + a^{j_2 j_1 k_1} - a^{k_1 j_2 j_1} - a^{j_2 k_1 j_1} \sim [1,1]$

$$\square\square\square \oplus \begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix} \oplus \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} = 10 \oplus 8 \text{ (twice)} \oplus 1$$

How do we see 8 appearing twice?

Tensor product decomposition algorithm (straightforward but long & boring proof)

Label boxes in simplest tableau in the product with ~~an~~ "a"s in the first row and "b"s in the second.

Add "a" boxes to other tableau in all legitimate ways, but without two (symmetrized) "a" boxes in (antisymmetrized) columns

Now add "b" boxes with no two "b"s in the same column,

$n_a \geq n_b$ and same # or more "a"s than "b"s reading right to left, then top to bottom

Example: $3 \otimes 3 = \square \otimes \boxed{a} = \square\boxed{a} \oplus \begin{smallmatrix} \square \\ \boxed{a} \end{smallmatrix} = 6 \oplus \bar{3}$

Example: $3 \otimes \bar{3} = \square \otimes \begin{smallmatrix} \boxed{a} \\ \boxed{b} \end{smallmatrix} - \begin{smallmatrix} \square \\ \square \end{smallmatrix} \otimes \boxed{a} = \begin{smallmatrix} \square & \boxed{a} \\ \square & \end{smallmatrix} \oplus \begin{smallmatrix} \square \\ \square \\ \boxed{a} \end{smallmatrix} = 8 \oplus 1$

$$\left\{ \begin{array}{l} \square\boxed{a} \to \begin{smallmatrix} \square & \boxed{a} \\ \boxed{b} & \end{smallmatrix} = 8 \\ \begin{smallmatrix} \square \\ \boxed{a} \end{smallmatrix} \to \begin{smallmatrix} \square \\ \boxed{a} \\ \boxed{b} \end{smallmatrix} = 1 \end{array} \right\} = 8 \oplus 1$$

(other tableaux break rule $n_a \geq n_b$)

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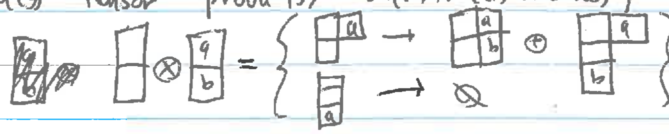
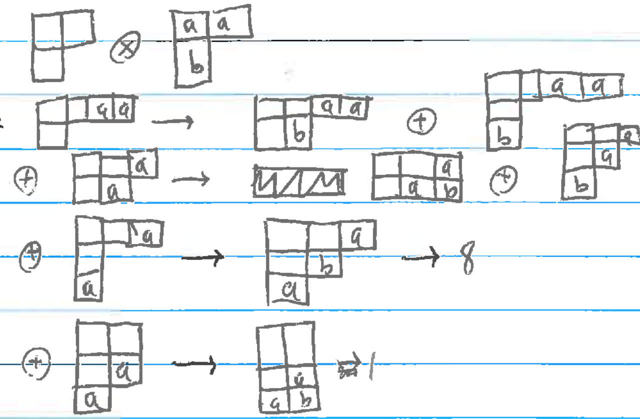
Young tableaux SU(3) tensor products, $SU(2) \times U(1) \subset SU(3)$, $U(3)$

Example: $\bar{3} \otimes \bar{3}$ =  $= \bar{6} + 3 = \overline{(3 \otimes 3)}$

Example: $8 \otimes 8$ =



$\to 27 \oplus 10$

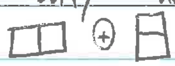
$\to \overline{10} \oplus 8$

$\to 8$

$\to 1$

So $8 \otimes 8 = 27 \oplus 10 \oplus \overline{10} \oplus 8 \oplus 8 \oplus 1$

For SU(2), only have upper indices in the first place

$a^{ij} \to$  $= \{1\} + \{0\}$ since ϵ^{ij} now ~~only~~ invariant

Application: see how SU(3) irreps transform under SU(2) (isospin) & U(1) (hyper-charge)

Consider 3 of SU(3): u & d quarks in isodoublet, s isosinglet

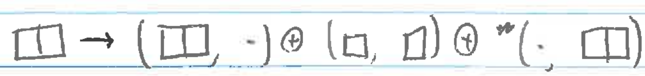
$\underset{1\ \ 2}{(u,d)} = 2_{1/3}$ $\quad \underset{3}{s} = 1_{-2/3}$ $\quad$ notation: $(2I+1)_Y$

Any SU(3) irrep corresponds to a Young tableau with n boxes

Any 3 index in tensor corresponds to isosinglet, 1 or 2 ~ isodoublet

Determine isospin & hypercharge from assignments of 1,2,3 labels to boxes in Young tableau

Convenient to split boxes into two groups, one with 1,2 indices, one with 3 indices

Example: 6 of SU(3) is 

$= 3_{2/3} \oplus 2_{-1/3} \oplus 1_{-4/3}$

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Return to 3d harmonic oscillator: what happened when we moved from U(3) to SU(3)? $U = e^{i\alpha_a T_a}$, no longer require $\begin{cases} \det U = 1 \\ \mathrm{Tr}\, T_a = 0 \end{cases}$

$U(3)$, $SU(2)\times U(1) \subset SU(3)$, More general groups $SU(N)$ 4/14/11

Breaking up $[T_a]^i{}_j = \{[T_a]^i{}_j - \frac{1}{3}\delta^i{}_j [T_a]^k{}_k\} + \frac{1}{3}\delta^i{}_j [T_a]^k{}_k \sim 8 \oplus 1$

Calling this ninth generator T_0, we see $[T_0, T_a] = 0$

So it is an abelian invariant subalgebra

By Schur's lemma, $T_0 \propto \mathbb{1}$ on any $SU(3)$ irrep

So $U = e^{i\theta}$ is just a phase

Relate $U(3)$ group multiplication to $SU(3)$ group multiplication

$U_1 = e^{i\theta_1} V_1 \quad U_2 = e^{i\theta_2} V_2 \Rightarrow U_1 U_2 = e^{i(\theta_1+\theta_2)} V_1 V_2 = e^{i(\theta_1+\theta_2)} V_3 = U_3$

Extra group is additive group on the circle (abelian $\theta_1+\theta_2 = \theta_2+\theta_1$): $U(1)$

Conclude $U(3) = SU(3)\times U(1)$ (no connection between $SU(3)$ & $U(1)$)

The $U(1)$ corresponds to an extra conserved charge

For the 3d harmonic oscillator, $Q_0 = a_k^\dagger a_k$ counts number n of excitations

More generally, any non-simple group can be written as a product of (semi-)simple groups and $U(1)$ factors

Returning to $SU(2)\times U(1) \subset SU(3)$, observe that $SU(2)\times U(1)$ can be generated by T_1, T_2, T_3, T_8 in $SU(3)$

Example: $\bar{3}$ of $SU(3)$ is $\yng(1,1) \rightarrow (\yng(1,1), \cdot) \oplus (\yng(1), \yng(1)) = 1_{2/3} \oplus 2_{-1/3}$

Example: 27 of $SU(3)$ is $\yng(4,2)$ NB remaining boxes

$\rightarrow (\yng(4,2), \cdot) \oplus (\yng(3,2), \yng(1)) \oplus (\yng(4,1), \yng(1)) \quad (3_2 \oplus 2_1 \oplus 4_1)$

$\oplus (\yng(2,2), \yng(2)) \oplus (\yng(3,1), \yng(2)) \oplus (\yng(4), \yng(2)) \quad (1_0 \oplus 3_0 \oplus 5_0)$

$\oplus (\yng(2,1), \yng(3)) \oplus (\yng(3), \yng(3)) \quad (2_{-1} \oplus 4_{-1})$

$\oplus (\yng(2), \yng(4)) \quad (3_{-2})$

All others would involve column on left from original $SU(3)$ irrep

Total: 27 states ✓ Can see that weights form a hexagon

Now generalize to $SU(N)$: same procedure, just need to track weights/indices

Start with defining rep of special unitary $N\times N$ matrices, generated by hermitian traceless $T_a = T_a^\dagger$ with $\mathrm{Tr}\, T_a = 0$

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SU(N) defining rep. weights and roots, Tensors for SU(N)

Dimension of defining rep of SU(N) is $\dim[su(N)] = N^2-1$

Rank is $N-1$ (could see from basis with diagonal & off-diagonal matrices)

Construct a basis in analogy to the Cell-Mann matrices

Normalization: $\mathrm{Tr}[T_a T_b] = \frac{1}{2}\delta_{ab}$

$m = N-1$ — Cartan subalgebra ($N-1$ elements): $[H_m]_{ij} = (2m(m+1))^{-1/2}\left[\sum_{k=1}^{m} \delta_{ik}\delta_{jk} - m\,\delta_{i,m+1}\delta_{j,m+1}\right]$

Now determine the weights and from them the roots

From $\binom{i}{j}$ basis, $N-1$ components — N weight vectors $\vec{v}_j$ with components $[\vec{v}_j]_m = [H_m]_{jj} = (2m(m+1))^{-1/2}\left[\sum_{k=1}^{m}\delta_{jk} - m\,\delta_{j,m+1}\right]$

Interested in lengths of roots and angles between them

Start with lengths of weights and angles between them

$$\vec{v}_j\cdot\vec{v}_j = \sum_{m=1}^{N-1}(2m(m+1))^{-1}\left[\sum_{k=1}^{m}\delta_{jk} - m\,\delta_{j,m+1}\right]^2 = \sum_{m=1}^{N-1}(2m(m+1))^{-1}\left[\sum_{k=1}^{m}\delta_{jk} + m^2\,\delta_{j,m+1}\right]$$

$m=j-1$

$$= \sum_{m=j}^{N-1}(2m(m+1))^{-1} + \frac{1}{2}\frac{(j-1)^2}{j(j-1)} = \frac{1}{2}\left[\frac{1}{j} - \frac{1}{N} + \frac{j-1}{j}\right] = \frac{N-1}{2N}$$

$\hookrightarrow \frac{1}{m} - \frac{1}{m+1}$ sum cancels in pairs

Similarly, $\vec{v}_i\cdot\vec{v}_j = -\frac{1}{2N}$ for $i\neq j \Rightarrow \vec{v}_i\cdot\vec{v}_j = \frac{1}{2}\delta_{ij} - \frac{1}{2N}$

To determine simple roots, need definition of positivity

More convenient to define positivity as the last nonzero component is > 0

Then $\vec{v}_1 > \vec{v}_2 > \cdots > \vec{v}_N$

Roots are of the form $\vec{v}_i - \vec{v}_j$ ($i\neq j$) positive when $i<j$

(Check number of roots $N^2-1-(N-1)$ consistent with choice of i & j)

$N-1$ simple roots are $\alpha_i = \vec{v}_i - \vec{v}_{i+1}$ (clearly not sum of others)

Check $\vec{\alpha}_i\cdot\vec{\alpha}_i = 1$ and $\vec{\alpha}_i\cdot\vec{\alpha}_j = -\frac{1}{2}\delta_{i,j\pm1}$ (for $i\neq j$) generally: $\vec{\alpha}_i\cdot\vec{\alpha}_j = \delta_{ij} - \frac{1}{2}\delta_{i,j-1} - \frac{1}{2}\delta_{i,j+1}$

$\therefore$ Dynkin diagram is $\underset{\alpha_1}{\bigcirc} - \underset{\alpha_2}{\bigcirc} - \underset{\alpha_3}{\bigcirc} - \cdots - \underset{\alpha_{N-1}}{\bigcirc}$

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Fundamental weights are $\vec{\mu}_j = \sum_{k=1}^{j} \vec{v}_k$

Check $2\,\vec{\alpha}_i\cdot\vec{\mu}_j/|\alpha_i|^2 = 2\,(\vec{v}_i - \vec{v}_{i+1})\cdot\left(\sum_{k=1}^{j}\vec{v}_k\right) = \sum_{k=1}^{j}(\delta_{ik} - \overset{\delta_{i,k-1}}{\delta_{i+1,k}}) = \delta_{ij}$

Tensors will help systematize SU(N) language, if we use invariant tensors to write everything in terms of upper indices.

That is, use Young tableaux

SU(N) Young tableaux, Tensor product, Dimensions 4/19/11

Let $|i\rangle$ with $i=1,\cdots N$ be the states of the defining rep

N-dimensional rep of $SU(N)$.

Let $A^{[i_1\cdots i_m]}$ be the totally antisymmetric part of $|i_1\rangle\otimes\cdots\otimes|i_m\rangle$

Claim that $A^{[i_1\cdots i_m]}$ is an irrep of $SU(N)$

Proof requires showing that the action of the generators takes this state to all others without changing the antisymmetry

Follows from thinking of generators as matrices with a single non-zero off-diagonal element

The highest weight of this irrep is $\sum_{k=1}^{m} \nu_k = \mu_m$ with $m<N$

For $m=N$, only one state (trivial rep) analogous to $\varepsilon^{i_1\cdots i_m}$

For $m>N$, antisymmetric combination vanishes. So this gives all fund. irreps

Correspond to Young tableaux with m boxes in a column

An arbitrary irrep has highest weight $\sum_{k=1}^{N-1} q_k \vec{\mu}_k = \boxed{[q_1, q_2, \cdots q_k]}$

Young tableau starts with q_{N-1} columns of $N-1$ boxes

then q_{N-2} columns of $N-2$ boxes, etc.

Highest weight tableau has 1 in every box on the first row

then n in every box on the nth row, etc.

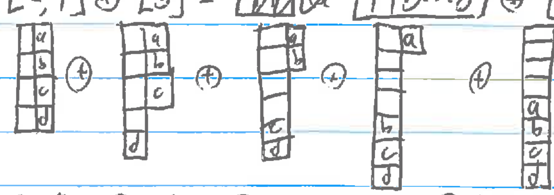
Yet another notation: write how many boxes are in each column $[5,3,2]$

Call the fundamental rep with highest weight μ_j $D^j = [j] = \boxed{0\cdots 1\cdots 0}$

↑ jth entry

Example:  $= D^2\otimes D^1 = [2]\otimes[1]$

$=$  $= [2,1]\oplus[3] =$ $\boxed{110\cdots0}\oplus\boxed{0010\cdots0}$

Example:  $= [4]\otimes[4] =$ 

$= [4,4]\oplus[5,3]\oplus[6,2]\oplus[7,1]\oplus[8]$ (note N-ality)

Rule generalizes to $\#_a \geq \#_b \geq \#_c \cdots$

Simple algorithm (with unpleasant proof) for calculating dimensions of irreps

"Factors over hooks" introduced with S_N

SU(N) irrep dimensionality, Complex conjugate reps, Subgroups 4/19/11

Factors over hooks algorithm:

1) Put N in the upper-left box for SU(N)
2) Label other boxes increasing by 1 in each column to the right, decreasing by 1 in each row below
3) Define F as the product of all factors in the boxes
4) Form a hook for each box (lines running down and to the right) and count the number of boxes the hook passes through (h)
5) Define H as the product of all h

Then the dimension of the irrep is F/H.

Example: $\dim(\square) = N$ since $F = \prod \boxed{N} = N$, $H = \prod \boxed{+} = 1$

Example: $\begin{array}{|c|c|}\hline N & N+1 \\ \hline N-1 & \\ \cline{1-1}\end{array} \Rightarrow F = (N)(N+1)(N-1)$

Hooks:  $= 3$ $\Big\}$ $\dim(\text{⊞}) = \frac{N(N+1)(N-1)}{3}$

check 8 for $N=3$ ✓

Most SU(N) reps are complex

Weights of complex conjugate reps are negative the weights of original rep

Lowest weight of defining rep is $\vec{\nu}_N + -\vec{\nu}_1$

So $-\vec{\nu}_N$ is highest weight of $\overline{D'}$

Since Cartan generators are traceless, $\sum_{k=1}^{N} \vec{\nu}_k = 0 \Rightarrow -\vec{\nu}_N = \sum_{k=1}^{N-1} \vec{\nu}_k = \vec{\mu}_{N-1}$

That is, $\overline{[1]} = [N-1]$

More generally, $\overline{[m]} = [N-m]$ or $\overline{[l_1, \dots l_k]} = [N-l_k, N-l_{k-1}, \cdots N-l_1]$

(reverse order since $l_1 > l_2 > \cdots > l_k$)

heck

Can immediately see (from dimensions) that $[N-1] \otimes [1] =$ adjoint + singlet

Tells us that $[N-1, 1]$ is the adjoint rep

Can also carry over decomposition into subgroups through Young tableaux

Consider $SU(N+M) \supset SU(N) \times SU(M) \times U(1)$ (think of block-diagonal $(N+M)\times(N+M)$ matrices)

The U(1) corresponds to a diagonal matrix with N Ms and M $-N$s

($\Rightarrow$ hermitian traceless)