Comments on BRST and the β function (21 September 2012)

Diagrams I didn't review Feynman rules and loop diagram calculations – been there, done that.

Transversity Srednicki makes an interesting comment below Eqn. 71.7 that for abelian fields we could simply exclude components of the gauge field A_μ parallel to k_μ in order to remove the gauge redundancy from the path integral. This is not possible for non-abelian gauge theories since $\delta A_\mu^a = -D_\mu^{ab}\theta^b(x)$ instead of $-\partial_\mu^{ab}\theta^b(x)$, where $\theta^b(x)$ parameterizes the infinitesimal non-abelian gauge transformation δ.

Ward identities Srednicki mentions (at the end of chapter 68) that BRST symmetry can be used to derive the Ward identities, citing Ramond's first field theory book (which I own and glanced through). He mentions the analogous Slavnov–Taylor identities at the start of chapter 74, but doesn't undertake the "lengthy project" of deriving them. In addition to Ramond's discussion (which focuses on abelian gauge theory), the first three chapters (15–17) of Weinberg2 use BRST symmetry to prove the renormalizability of non-abelian gauge theories (which I also only glanced through).

BRST I liked the way Srednicki introduces the BRST transformation δ_B by making the infinitesimal gauge transformation parameter proportional to the ghost field, so that gauge invariance $\Longrightarrow$ BRST invariance. He then further restricts the transformation by requiring that it be nilpotent ($\delta_B(\delta_B\mathcal{O}) = 0$ or equivalently $Q_B^2 = 0$), which fixes the BRST transformation of the ghost field. Nilpotency turns out to be crucial for the determination of physical states (discussed below), but at this point I was best able to motivate this restriction by noting that it makes the modified lagrangian $\mathcal{L} = \mathcal{L} + \delta_B\mathcal{O}$ (Eqn. 74.17) automatically BRST-invariant. I wasn't able to motivate the anti-ghost BRST transformation (Eqn. 74.15) that introduces the (non-propagating) auxiliary field, but even this sketchy derivation of the R_ξ-gauge-fixed lagrangian in Eqn. 74.22 left me more comfortable than the corresponding section in Peskin & Schroeder, which uses this lagrangian as a starting point (Eqn. 16.44, related to Eqn. 16.34 by completing the square).

Symmetries Srednicki lists (below Eqn. 74.24) the various symmetries respected by the gauge-fixed Yang–Mills action. Most of these seem straightforward, but I wondered whether the anti-ghost translation invariance serves any purpose.

Physical states I Before our meeting, I wrote: "The physical states are those that are annihilated by the BRST charge

$$Q_B \equiv \int d^3x \sum_I \frac{\partial\mathcal{L}}{\partial\left(\partial_0\Phi_I(x)\right)}\delta_B\Phi_I(x)$$

"(where Φ_I stands for all the fields), but cannot be written in terms of Q_B acting on any other state. (This spells out the 'cohomology' jargon that Srednicki uses, which is Peskin & Schroeder's $\mathcal{H}_0$.) Srednicki motivates this restriction by noting that states satisfying it continue to do so after time evolution (due to the BRST symmetry of the hamiltonian, $[H, Q_B] = 0$), and it kills off the unwanted ghosts and longitudinal gauge polarizations. Furthermore, the identification of physical states that differ by $Q_B\,|\xi\rangle$ removes the gauge redundancy. Weinberg2 includes a similar discussion."

Physical states II During our meeting, Oliver emphasized that cohomology is a generically important concept for any nilpotent operator. To review the definitions, *closed* states $|\psi\rangle$ in the **kernel** of Q_B are annihilated by $Q_B|\psi\rangle = 0$; *exact* states $|\zeta\rangle$ in the **image** of Q_B are obtained by acting on some other state $|\zeta\rangle = Q_B|\xi\rangle$; and the **cohomology** consists of closed states that are not exact (states that are in the kernel but not in the image). States in the cohomology are only distinct if their difference is also not exact; otherwise they are identified. Since exact states have zero norm ($\langle\zeta|\zeta\rangle = \langle\xi|Q_B^2|\xi\rangle = 0$),[1] we can rephrase this to say that the cohomology consists of closed states with non-zero norm, and the difference between distinct states in the cohomology is also non-zero.

Physical states III Also during our meeting, Oliver provided an intuitive way to think about this identification of physical states, by noting that the BRST transformation is related to the gauge transformation: We want states to be gauge-invariant ($\Longrightarrow$ closed) and also distinct in the sense that two such states cannot be related by a gauge transformation ($\Longrightarrow$ their difference is not exact).

Fixed points After dealing with BRST we skipped straight to the β function, first considering its sign and the possibility that the first two terms in its series expansion can have opposite signs, thereby producing a non-trivial fixed point (i.e., a fixed point located at non-zero coupling $g > 0$). For SU(N) gauge theories, this leads to my friend the conformal window. Of course, if the coupling is large enough that higher-order terms in the series become important, one might question the meaningfulness of this perturbative series itself. Oliver reminded us of the Banks–Zaks calculation, which performed a systematic expansion around the weak-coupling infrared-attractive fixed points identified by Caswell from the two-loop β function. Banks & Zaks expand in $(N_f^{(af)} - N_f)$, where $N_f^{(af)}$ is the number of fermions for which the first term in the β function changes sign and the theory loses asymptotic freedom. When $(N_f^{(af)} - N_f)$ is small, the IR fixed point appears at weak enough coupling (in your scheme of choice) that perturbation theory should be reliable.[2]

Schemes I mentioned that only the first two terms in the series expansion of $\beta(g)$ are scheme-independent,[3] and we recalled the facts that the existence of an IR fixed point is scheme-independent, while the value of the coupling at which it appears is scheme-dependent. Oscar asked about the physical meaning of a scheme transformation, and we reached a consensus that scheme transformations amount to redefinitions of the coupling.

Gribov! An issue that came to my mind in the context of gauge fixing is the "residual, discrete gauge ambiguity discovered by Gribov (1978), which has never caused [Tom] trouble." I have been told that this "obstruction to a consistent [non-abelian] gauge fixing procedure is topological in origin", and I've seen (what seem like) lots of hep-lat papers that calculate gluon and ghost propagators in either Lorentz or Coulomb gauge and try to relate their IR properties to considerations of "Gribov copies", "Gribov horizons", or "the refined Gribov–Zwanziger scenario". My suspicion is that this is not an issue that would be worthwhile for us to discuss, though I meant (and forgot) to bring it up during our meeting.

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[1] This relies on the hermiticity of Q_B, which according to Srednicki (below Eqn. 74.26) follows from treating both the ghost and anti-ghost as independent hermitian fields.

[2] Both Caswell and Banks & Zaks consider generic SU(N) in addition to SU(3); I mistakenly said "Callan" instead of Caswell during our meeting.

[3] I have posted solutions to the problem of showing that the first two terms in the series expansion of $\beta(g)$ are scheme-independent, and it looks like the same approach will show that the third term is not.