Lattice Simulations of Nonperturbative Quantum Field Theories

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Final Thesis Talk
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Outline

- Lattice simulations
  - We did this last time
- Quantum field theory
  - A very very very brief introduction
- Phase transitions
  - Picking up where we left off in December
- Solitons
  - Time permitting

Lattice Simulations

Lattice Simulations

- We did this last time

Lattice Simulations

- We did this last time
- So we'll take it for granted that we have (Markov chain Monte Carlo) algorithms that will reliably and efficiently reproduce the Boltzmann distribution
- Allowing us to simulate statistical systems on the computer

Quantum Field Theory

Quantum Field Theory

(in five minutes)

Quantum Field Theory

(in five minutes) (or less)

Quantum Field Theory

(in five minutes) (or less)

- Combination of special relativity and quantum mechanics

- Free particle Schrödinger equation:

$$E=\frac{p^2}{2m} \qquad E\rightarrow i\frac{\partial}{\partial t} \qquad p\rightarrow i\vec{\nabla}$$

$$i\frac{\partial}{\partial t}\phi=\frac{-1}{2m}\vec{\nabla}^2\phi$$

- Relativistic analog is the Klein-Gordon equation:

$$E^2=p^2+m^2 \qquad \frac{\partial^2}{\partial t^2}\phi-\vec{\nabla}^2\phi+m^2\phi=(\partial^2+m^2)\phi=0$$

Quantum Field Theory

(in five minutes) (or less)

- Klein-Gordon equation only really makes sense if ϕ is treated as a field
- Otherwise there are unbounded negative-energy solutions, and negative probability densities
- For example,

$$\phi(x)=\int\frac{d^4k}{(2\pi)^4}[a(k)e^{-ik\cdot x}+a^\dagger(k)e^{ik\cdot x}]$$

ϕ^4 Quantum Field Theory

- Lagrangian: $$L=\frac{1}{2}(\partial_\alpha \phi)^2-\frac{1}{2}\mu_0^2\phi^2-\frac{\lambda}{4}\phi^4$$

- Equation of motion: $$(\partial^2+\mu_0^2)\phi=-\lambda\phi^3$$

  Same form as Klein-Gordon equation, only nonlinear

- Obvious constant solutions:

$$\phi=0 \qquad \phi=\pm\sqrt{\frac{-\mu_0^2}{\lambda}}$$

ϕ^4 Quantum Field Theory Phases

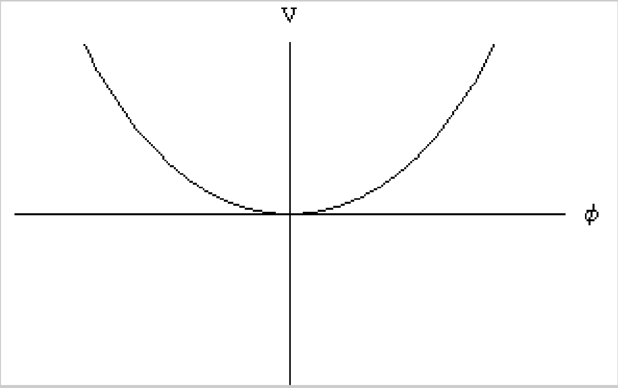
- Lagrangian: $L = \frac{1}{2}(\partial_\alpha \phi)^2 - \frac{1}{2}\mu_0^2 \phi^2 - \frac{\lambda}{4}\phi^4$
- Equation of motion: $(\partial^2 + \mu_0^2)\phi = -\lambda \phi^3$

  Same form as Klein-Gordon equation, only nonlinear

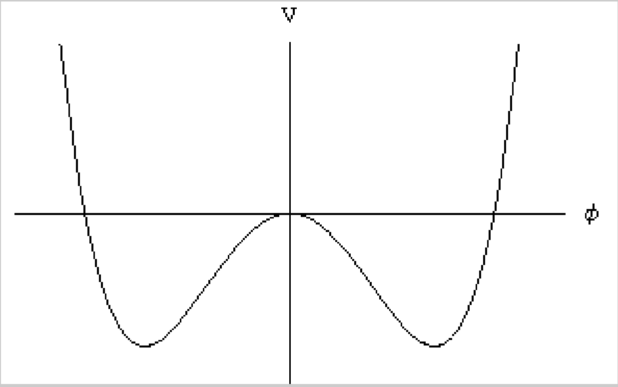
- Two phases:

$\langle \phi \rangle = 0$

$\langle \phi \rangle = \pm\sqrt{\frac{-\mu_0^2}{\lambda}}$



Symmetric phase

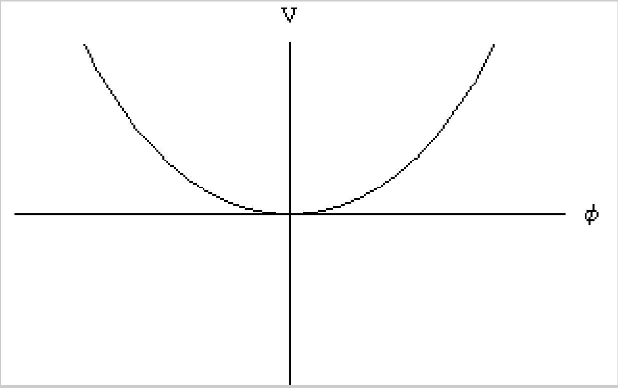


Broken phase

ϕ^4 Quantum Field Theory Phases

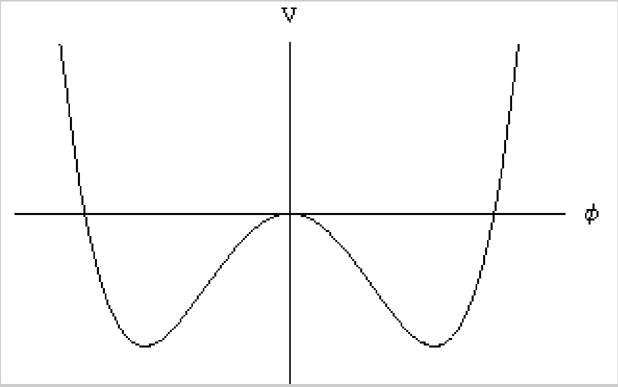
- Going from symmetric phase to broken phase breaks symmetry:
  - In symmetric phase values of ϕ are randomly +/- ("up-down symmetry")
  - In broken phase, values are either all + or all -

$$\langle \phi \rangle = 0$$



Symmetric phase

$$\langle \phi \rangle = \pm \sqrt{\frac{-\mu_0^2}{\lambda}}$$

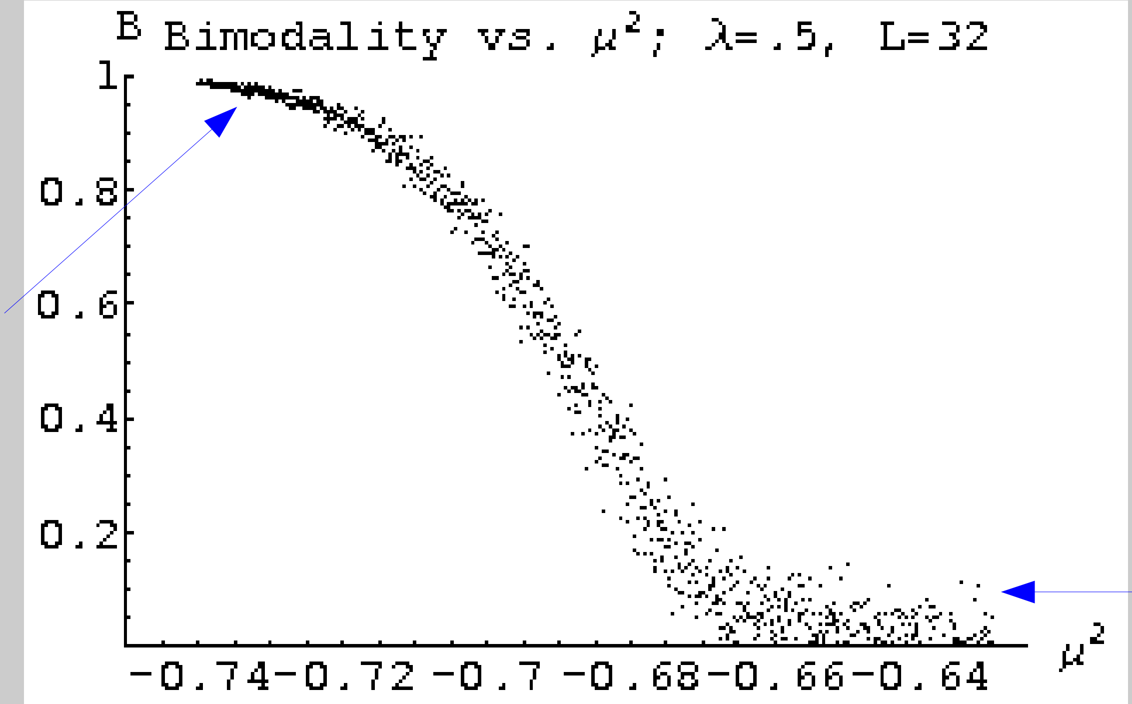


Broken phase

ϕ^4 Quantum Field Theory Phases

- So there's a phase transition!

Symmetric phase



Broken phase

- Let's put some lattice simulations on the cluster to calculate the critical point!

ϕ^4 Theory Simulations

- Wait, are we perhaps being a bit glib?

ϕ^4 Theory Simulations

- Wait, are we perhaps being a bit glib?
- Well, yes, but we don't need to be
- There exists a rigorous mapping from quantum field theories to classical statistical mechanics, through Wick rotation,

$$t = x_0 \rightarrow -i x_4 \qquad d^4 x = d x_0 d^3 x \rightarrow -i d^3 x \, d x_4 = -i d^4 x_E$$

$$|x|^2 = x_0^2 - \vec{x}^2 \rightarrow -(\vec{x}^2 + x_4^2) = -|x_E|^2$$

$$\partial^2 = \frac{\partial^2}{\partial x_0^2} - \vec{\nabla}^2 \rightarrow -\vec{\nabla}^2 - \frac{\partial^2}{\partial x_4^2} = -\partial_E^2$$

$$L = \frac{1}{2}(\partial_\alpha \phi)^2 - \frac{1}{2}\mu_0^2 \phi^2 - \frac{\lambda}{4}\phi^4 \rightarrow -\frac{1}{2}(\partial_{E\alpha}\phi)^2 - \frac{1}{2}\mu_0^2\phi^2 - \frac{\lambda}{4}\phi^4 = -L_E$$

ϕ^4 Theory Simulations

- Minkowski spacetime converted into four-dimensional Euclidean space
- The Euclidean Lagrangian L_E has the form of an energy density $L_E = \frac{1}{2}(\partial_{E\alpha}\phi)^2 + \frac{1}{2}\mu_0^2\phi^2 + \frac{\lambda}{4}\phi^4$
- The action S_E then has the form of an energy

$$S_E = \int d^4 x_E L_E$$

- And the Feynman path integral behaves just like a thermodynamic partition function

$$\int D\phi(x) e^{iS} = \int D\phi(x) e^{i\int d^4 x L} \rightarrow \int D\phi(x) e^{-\int d^4 x_E L_E} = \int D\phi(x) e^{-S_E}$$

ϕ^4 Theory Simulations

- So we can investigate the ϕ^4 quantum field theory by simulating the corresponding statistical system using the techniques discussed when last we met

- The discretized lattice action S_E is

$$S_E = -\sum_{\langle ij \rangle} \phi_i \phi_j + \sum_n [(d + \frac{\mu_{0L}^2}{2})\phi_n^2 + \frac{\lambda_L}{4}\phi_n^4]$$

where μ^2_{0L} and λ_L depend on the lattice spacing a

$$\mu_{0L}^2 = \mu_0^2 a^2$$
$$\lambda_L = \lambda a^2$$

ϕ^4 Theory Phases

$$S_E = -\sum_{\langle ij \rangle} \phi_i \phi_j + \sum_n [(d + \frac{\mu_{0L}^2}{2})\phi_n^2 + \frac{\lambda_L}{4}\phi_n^4]$$

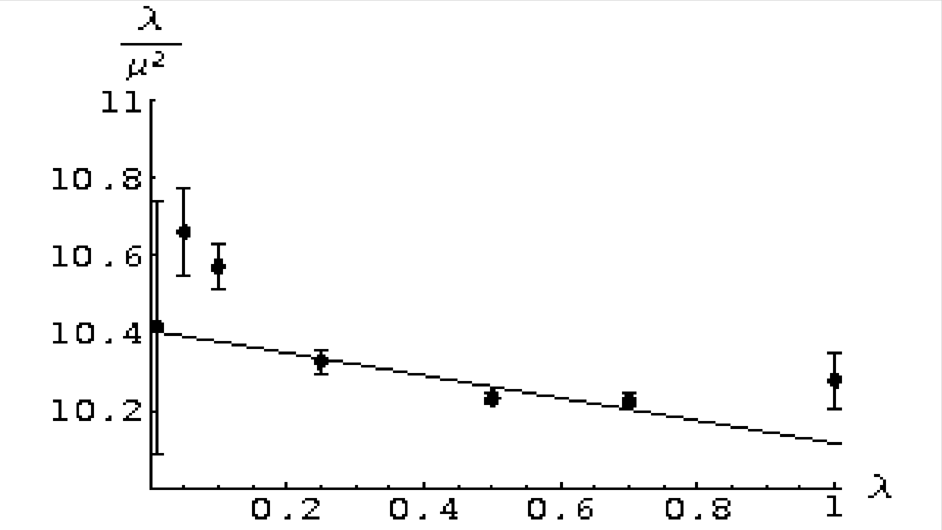
Since, we're interested in the continuum theory (a→0), we have a slight problem:

$$\lim_{a\to 0} \mu_{0L}^2 = \lim_{a\to 0} \mu_0^2 a^2 = 0$$
$$\lim_{a\to 0} \lambda_L = \lim_{a\to 0} \lambda a^2 = 0$$

which we solve by considering the critical coupling constant

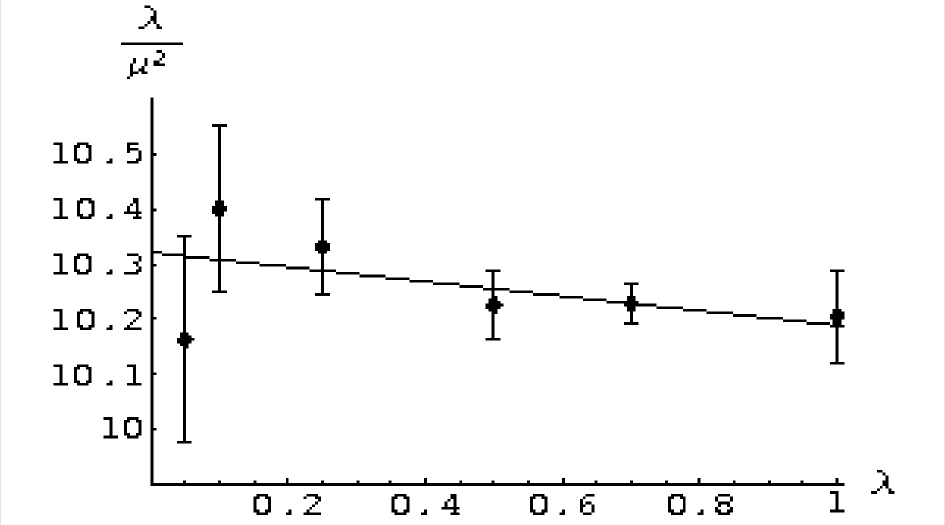
$$[\lambda/\mu^2]_{crit} = \lim_{a\to 0} [\lambda_L/\mu_L^2]_{crit}$$

Preliminary Phase Results



$$[\lambda/\mu^2]_{crit} = 10.27^{+.06}_{-.05}$$

Published Results:



$$[\lambda/\mu^2]_{crit} = 10.26^{+.08}_{-.04}$$

W. Loinaz & R. S. Willey, Phys. Rev. D. **58**, 076003 (1998).

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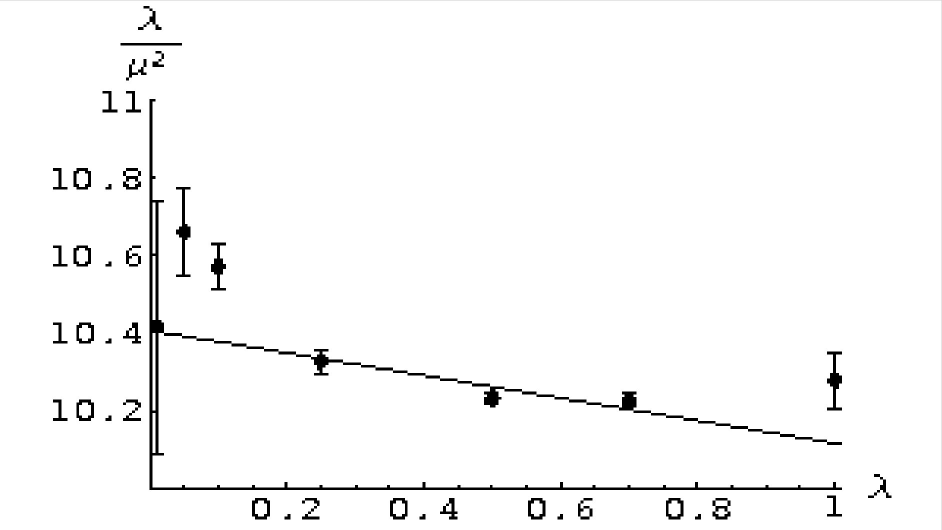

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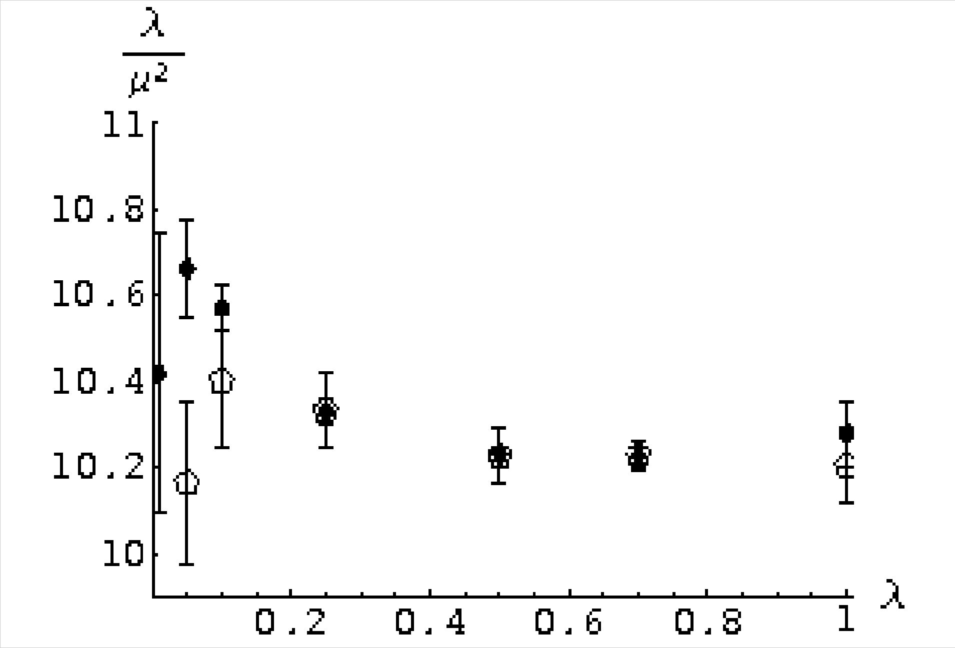

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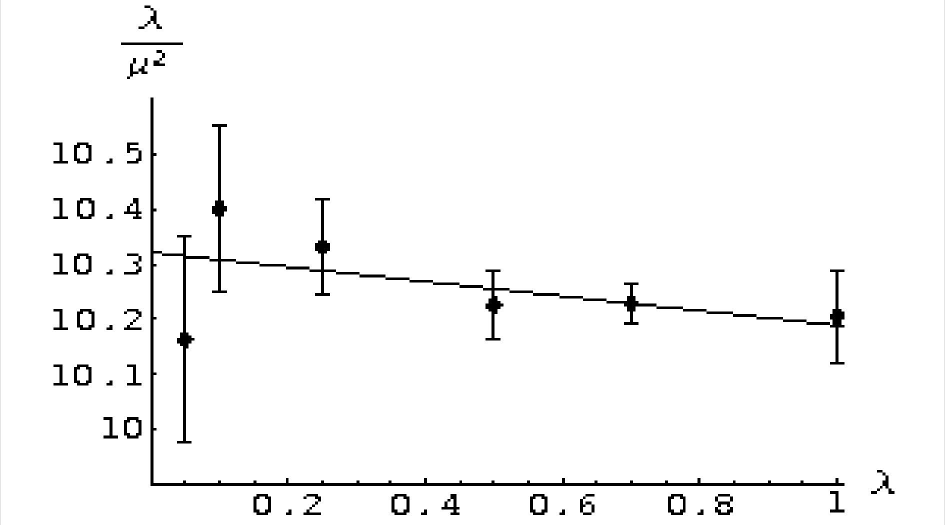

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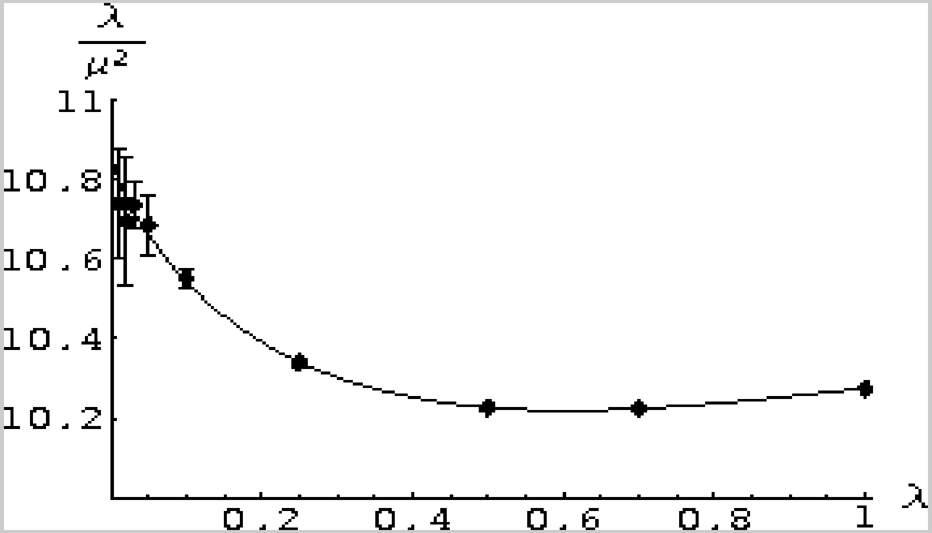
Final Phase Results

More data confirms higher-order effects:



Regression includes $\lambda \log[\lambda]$ and $\lambda^2 \log[\lambda]$ terms, reflecting some mixture of higher-order loop corrections and systematic effects introduced by approximations made during the discretization procedure

Final Phase Results

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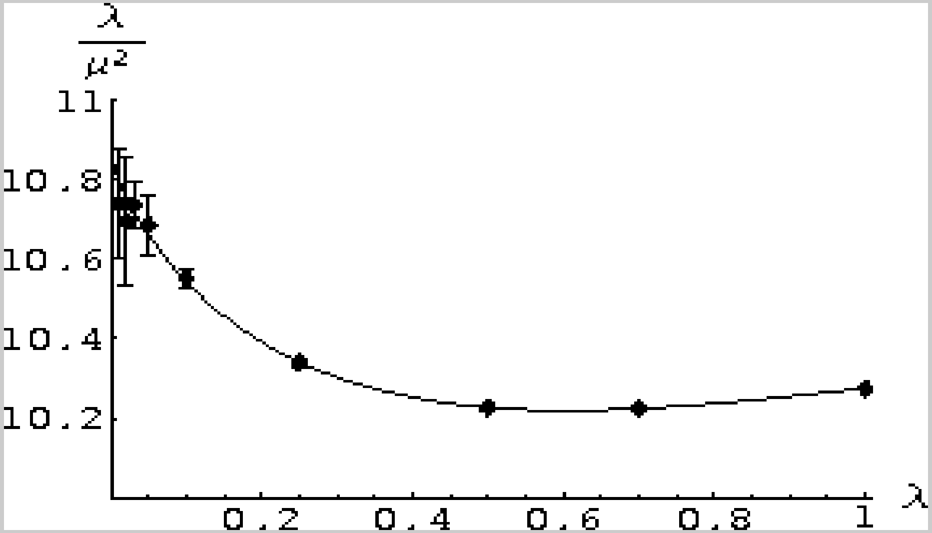
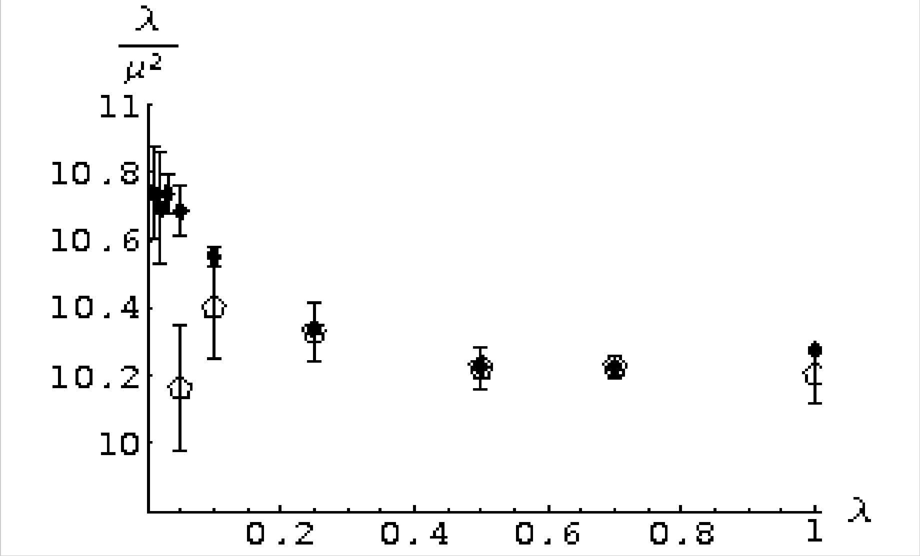
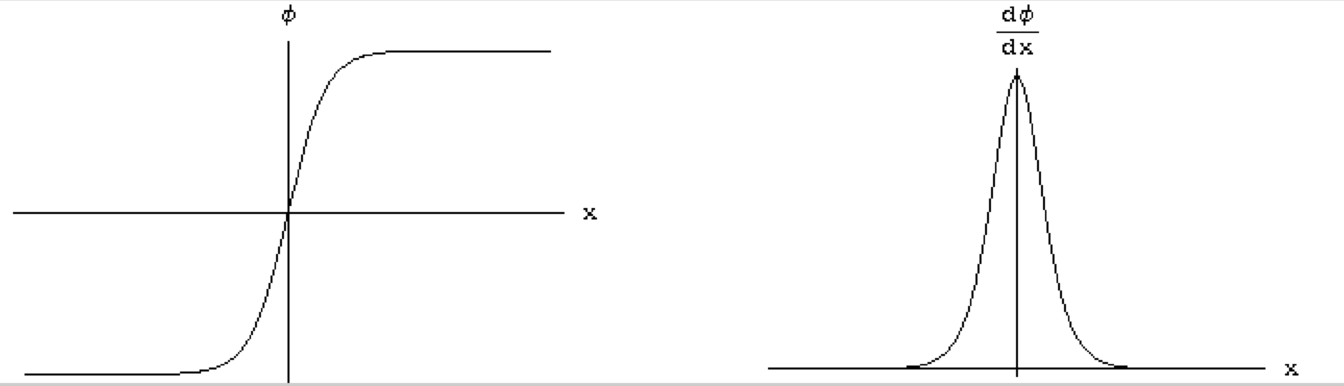




$$[\lambda/\mu^2]_{crit} = 10.85^{+.03}_{-.08}$$

ϕ^4 Theory Solitons

- Recall equation of motion: $(\partial^2 + \mu_0^2)\phi = -\lambda\phi^3$
- Nonlinearity allows for some even more interesting solutions $$\phi = \pm\sqrt{\frac{-\mu_0^2}{\lambda}}\tanh\left[x\sqrt{\frac{-\mu_0^2}{2}}\right]$$
- These 'kink' solutions continuously connect two degenerate ground states in broken phase:



ϕ^4 Theory Solitons

- Can show that these kink solutions are solitons, stable localized waves

- Energy (mass) of solitons can be determined by calculating energy of system with soliton and subtracting ground state energy

- Easy calculation classically: $\phi = \pm\sqrt{\frac{-\mu_0^2}{\lambda}}\tanh\left[x\sqrt{\frac{-\mu_0^2}{2}}\right]$

$$E = \int d^2x \left(\frac{1}{2}(\partial_\alpha \phi)^2 + \frac{1}{2}\mu_0^2\phi^2 + \frac{\lambda}{4}\phi^4 \right) = \frac{2\sqrt{2}}{3}\frac{\left(\sqrt{-\mu_0^2}\right)^3}{\lambda} = M_{sol}$$

ϕ^4 Theory Solitons

- Classically, $$M_{sol} = \frac{2\sqrt{2}}{3} \frac{\left(\sqrt{-\mu_0^2}\right)^3}{\lambda}$$

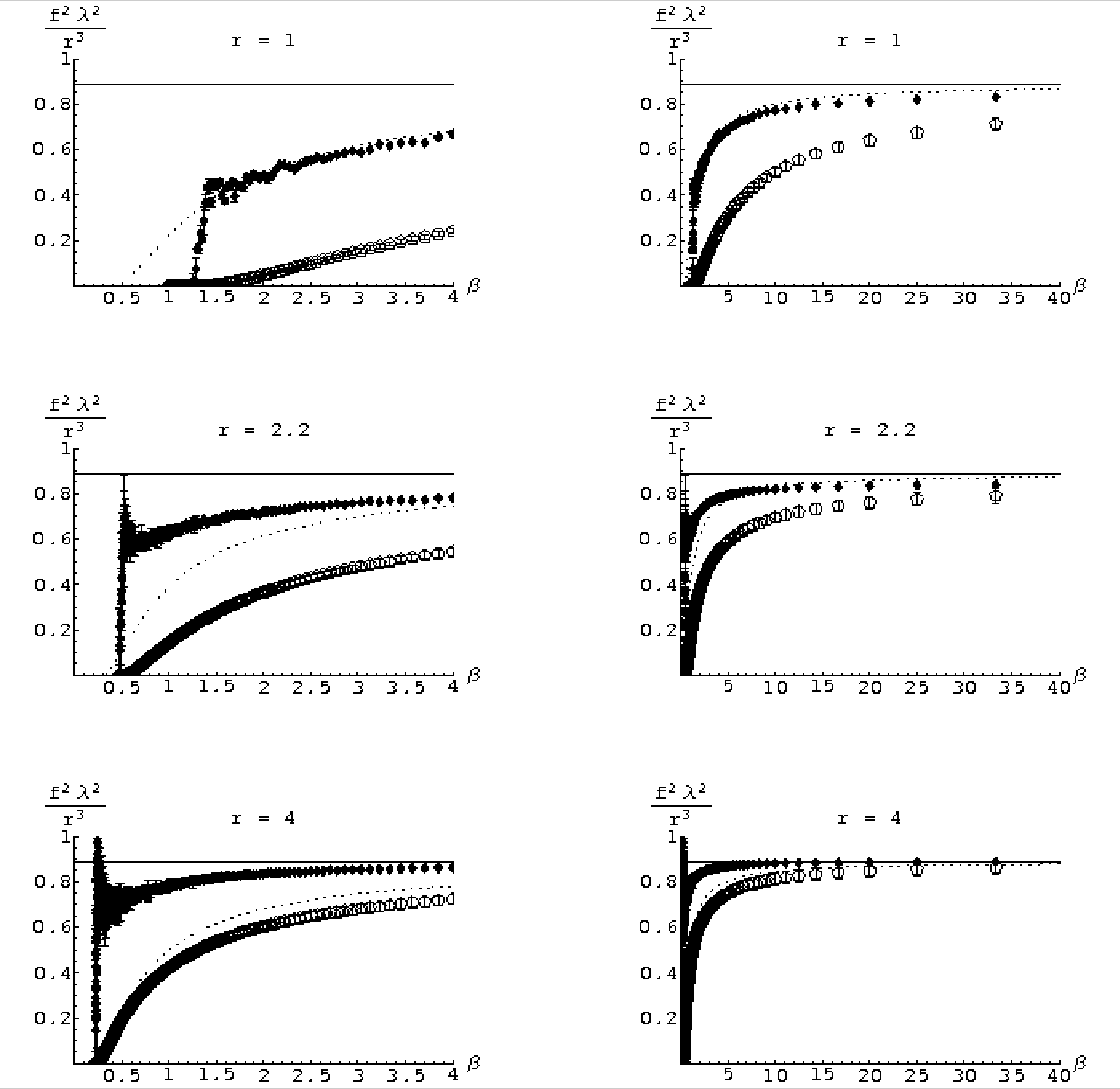
- Must take quantum effects into account. First-order (in ħ, zeroth-order in λ) approximation:

$$M_{sol} = \frac{2\sqrt{2}}{3} \frac{\left(\sqrt{-\mu_0^2}\right)^3}{\lambda} + \sqrt{-\mu_0^2}\left(\frac{1}{6}\sqrt{\frac{2}{3}} - \frac{3}{\pi\sqrt{2}}\right) + O(\lambda)$$

- What do the simulations say?

$r = -\mu_0^2$

$\beta = \frac{1}{\lambda}$



Classical limit →

Classical limit ↓

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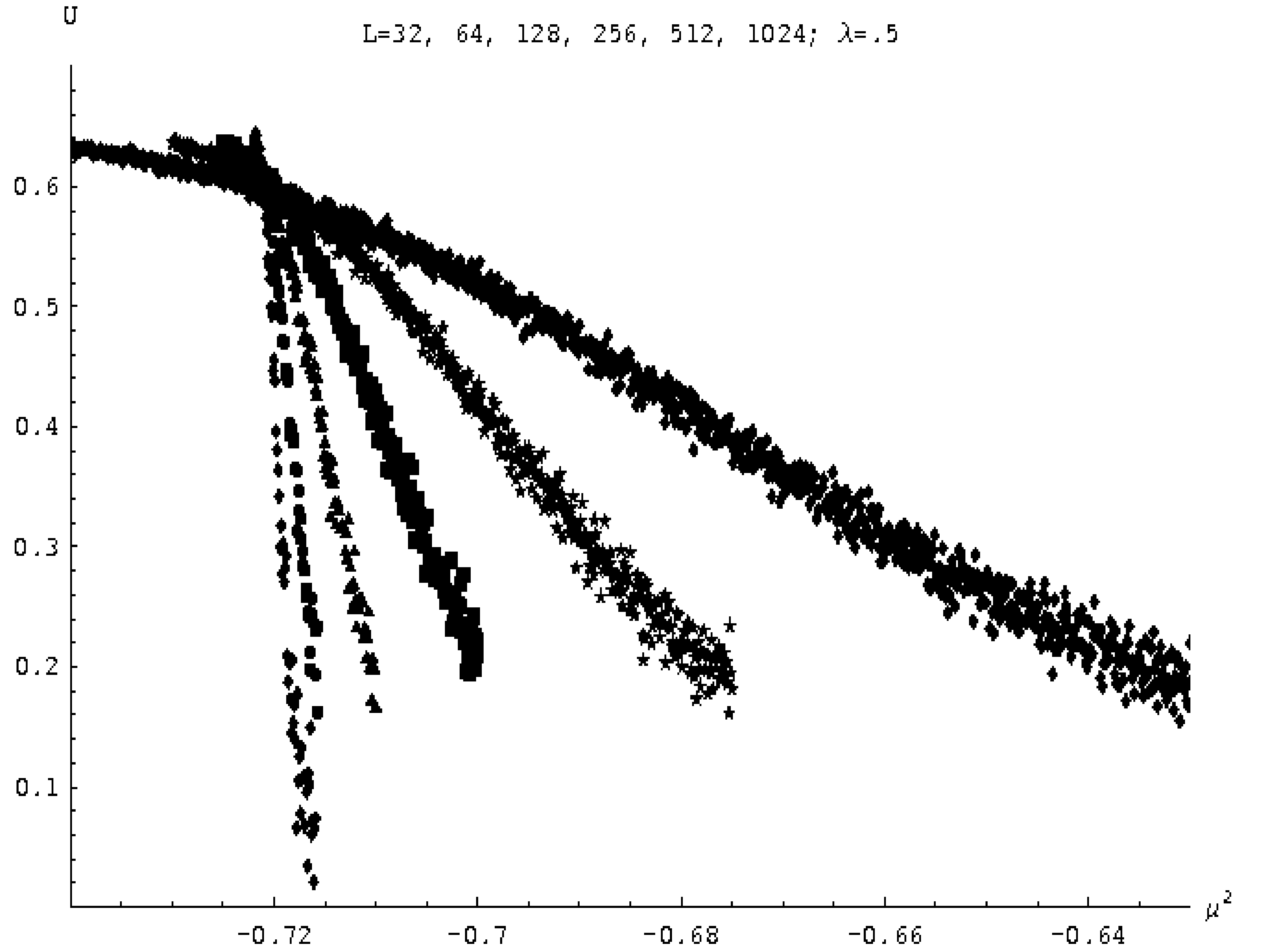
- Prof. Kaplan
- Chris Bednarzyk '01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L=32, 64, 128, 256, 512, 1024; λ=.5
0.6
0.5
0.4
0.3
0.2
0.1
-0.72
-0.7
-0.68
-0.66
-0.64
μ2