Lattice $\mathcal{N} = 4$ SYM

David Schaich, 13 October 2016

Goals of this informal and pedagogical presentation

- Focus on big-picture context and motivation
  (relatively little about my own contributions)
- Basic difficulties with supersymmetry on the lattice
- How we circumvent them in four-dimensional $\mathcal{N} = 4$ SYM
  (analogous lattice systems in 2d & 3d)
- **Entry point:** arXiv:1512.01137 **Review:** arXiv:0903.4881

Motivations / context for lattice supersymmetry

- **Theory: Symmetries** simplify systems $\longrightarrow$ analytic insights
  into confinement, dynamical symmetry breaking, conformality...
  Lattice is new non-perturb. method to explore / refine / extend insights
- **Dualities:** Same physics from theories with different fields & interactions
  Relate "electric" & "magnetic" gauge theories — Seiberg duality
  Relate gauge & gravity theories — AdS/CFT duality or "holography"
  Method: Conjecture & check (exploiting susy), may be extended by lattice
- **Pheno: BSM** is familiar context for susy-based model building
  Relies on (dynamical) spontaneous supersymmetry breaking $\longrightarrow$ lattice
  Speculate LHC constraints prefer non-perturbative new physics?
- **Modelling:** Attempts to study everything from QCD at finite density
  to non-Fermi liquids based on AdS/CFT holography
  Lattice could provide new input to these efforts — validate or refine

Lattice gauge theory in a nutshell

- Non-perturbative, gauge-invariant QFT regularization, directly in d dims
- Replace continuous spacetime with finite grid of discrete sites
  Work in euclidean space $\longrightarrow$ SO$(d)_{\text{euc}}$ rotations $\Lambda_{\mu\nu}$
- One of its drawbacks: Discretization breaks Poincaré invariance
  Equivalently, lattice spacing a between sites introduces UV cutoff a^{-1}
- Improves upon naive momentum cutoff by preserving hypercubic subgroup
  $\longrightarrow$ recover full Poincaré upon removing cutoff ($a \to 0$ continuum limit)

Naive obstacle to lattice supersymmetry

- Supersymmetries extend Poincaré spacetime symmetry
- Add spinorial generators Q_α^A and $\overline{Q}_{\dot\alpha}^A$ with $A = 1, \cdots, \mathcal{N}$
  Transform under global SU$(\mathcal{N})_R$ "R" symmetry

$$\left[Q_\alpha^A, P_\mu\right] = 0 \qquad\qquad \left[Q_\alpha^A, \Lambda_{\mu\nu}\right] \propto \frac{1}{4}\left[\gamma_\mu, \gamma_\nu\right]_\alpha^\beta Q_\beta^A$$

$$\left\{Q_\alpha^A, \overline{Q}_{\dot\alpha}^B\right\} = 2\delta^{AB}\sigma_{\alpha\dot\alpha}^\mu P_\mu \qquad\qquad \left(Q_\alpha^A\right)^2 = 0$$

  (Can recall $[P, P] \sim 0$; $\quad [P, \Lambda] \sim P$; $\quad [\Lambda, \Lambda] \sim \Lambda$;
  $\left\{Q_\alpha^A, Q_\beta^B\right\} \sim \epsilon_{AB} Z^{AB}$ central charge)

- **Lattice:** P_μ generates infinitesimal spacetime translations
  P_μ does not exist in discrete spacetime
  $\Longrightarrow$ explicit susy breaking at classical level of algebra
- **Consequence:** Relevant or marginal susy-violating operators
  (typically many) no longer forbidden and have to be fine-tuned
  Scalar mass and Yukawa terms make scalar fields especially problematic
  (squarks from matter multiplets or extended susy, $\mathcal{N} > 1$)
- **Special case:** Can preserve closed sub-algebra for $\mathcal{N} = 4$ SYM in 4d

$\mathcal{N}=4$ SYM in four dimensions as simplest QFT

- AdS / CFT; integrability / amplituhedron
- **Maximally supersymmetric:** Restricting to helicities ≤ 1,
  Q_α^A act as four 'lowering operators' on massless Clifford vacuum states
  [highest-weight state annihilated by all $\overline{Q}$ in frame $(E,0,0,E)$]

State	Helicity	Flavor $\text{SU}(4)_R$
$\|\Omega_1\rangle$	1	$\mathbf{1}$
$Q_\alpha^A \|\Omega_1\rangle$	1/2	$\mathbf{4}$
$Q_\beta^B\, Q_\alpha^A \|\Omega_1\rangle$	0	$\mathbf{6}$
$Q_\gamma^C\, Q_\beta^B\, Q_\alpha^A \|\Omega_1\rangle$	$-1/2$	$\overline{\mathbf{4}}$
$Q_\delta^D\, Q_\gamma^C\, Q_\beta^B\, Q_\alpha^A \|\Omega_1\rangle$	-1	$\mathbf{1}$

- **Yang–Mills:** Only single super-multiplet
  Contains the gauge field A_μ, four fermions Ψ^A and six scalars Φ^{AB}
  all massless and in adjoint rep. of $\text{SU}(N)$ gauge group
- **Action:** Usual kinetic, Yukawa, four-scalar terms; only param. is $\lambda = g^2 N$
- **Conformal:** $\beta(\lambda) = 0$ for all couplings (line of fixed points)

Topological twisting (equivalent construction from orbifolding)

- Expand 4×4 matrix of 16 supercharges in basis of γ matrices

$$\begin{pmatrix} Q_\alpha^1 & Q_\alpha^2 & Q_\alpha^3 & Q_\alpha^4 \\ \\ \overline{Q}_{\dot\alpha}^1 & \overline{Q}_{\dot\alpha}^2 & \overline{Q}_{\dot\alpha}^3 & \overline{Q}_{\dot\alpha}^4 \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu\gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu\gamma_\mu\gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

  Closed susy subalgebra $\{\mathcal{Q},\mathcal{Q}\} = 2\mathcal{Q}^2 = 0$ can be preserved on lattice

- **Observation:** Expansion mixes spacetime symmetry (along each column)
  and R symmetry (along each row)
  $\Longrightarrow$ Expanding in **integer-spin** reps of "twisted rotation group"

$$\text{SO}(4)_{tw} \equiv \text{diag}\Big[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R\Big] \qquad \text{with} \qquad \text{SO}(4)_R \subset \text{SO}(6)_R$$

- Simple change of variables in flat spacetime,
  replacing spinors with anti-symmetric tensors
- **Restriction:** Need at least 2^d supercharges for expansion
  Only applicable to $\mathcal{N}=4$ SYM in 4d (more possibilities in 2d & 3d)

Twisted fields and their transformations

- **Four fermions:** Majorana Ψ^A expand just like supercharges

$$\left(\begin{array}{cccc} \Psi^1 & \Psi^2 & \Psi^3 & \Psi^4 \end{array} \right) \longrightarrow \left(\eta,\ \psi_\mu,\ \chi_{\mu\nu},\ \overline{\psi}_\mu,\ \overline{\eta}\right)$$

- **Complication:** Only have $\mathrm{SO}(4)_R \subset \mathrm{SU}(4)_R \simeq \mathrm{SO}(6)_R$
  $\Longrightarrow$ Scalar fields in $\mathrm{SO}(6)_R$ vector rep $\Phi^A \longrightarrow (B_\mu,\ \phi,\ \overline{\phi})$

- **Solution:** Combine $4 + 6$ bosons in complexified gauge fields

$$\mathcal{A}_a = (A_\mu,\ \phi) + i(B_\mu,\ \overline{\phi}) \qquad\qquad \overline{\mathcal{A}}_a = (A_\mu,\ \phi) - i(B_\mu,\ \overline{\phi})$$

  Similarly combine $\psi_a = (\psi_\mu,\ \overline{\eta})$ and $\chi_{ab} = (\chi_{\mu\nu},\ \overline{\psi}_\mu)$

$$\mathcal{Q}_a = (\mathcal{Q}_\mu,\ \overline{\mathcal{Q}}) \text{ and } \mathcal{Q}_{ab} = (\mathcal{Q}_{\mu\nu},\ \overline{\mathcal{Q}}_\mu)$$

- **$\mathcal{Q}$ transformations:** Nilpotent ($\mathcal{Q}^2 = 0$), exchanges bosons $\longleftrightarrow$ fermions

$$\begin{aligned} &\mathcal{Q}\ \mathcal{A}_a = \psi_a && \mathcal{Q}\ \psi_a = 0 \\ &\mathcal{Q}\ \chi_{ab} = -\overline{\mathcal{F}}_{ab} && \mathcal{Q}\ \overline{\mathcal{A}}_a = 0 \\ &\mathcal{Q}\ \eta = d && \mathcal{Q}\ d = 0 \end{aligned}$$

  d is bosonic auxiliary field for off-shell susy, with standard e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{U}_a$

- **Discretize:** Simply replace $\mathcal{A}_a \longrightarrow \mathcal{U}_a$ above,
  note geometric site / link / plaq. structure from lattice gauge trans.

$$\begin{aligned} \mathcal{U}_a(n) &\to G(n)\mathcal{U}_a(n)G^\dagger(n+\widehat{\mu}_a) & \psi_a(n) &\to G(n)\psi_a(n)G^\dagger(n+\widehat{\mu}_a) \\ \overline{\mathcal{U}}_a(n) &\to G(n+\widehat{\mu}_a)\overline{\mathcal{U}}_a(n)G^\dagger(n) & \eta(n) &\to G(n)\eta(n)G^\dagger(n) \\ & & \chi_{ab}(n) &\to G(n+\widehat{\mu}_a+\widehat{\mu}_b)\chi_{ab}(n)G^\dagger(n) \end{aligned}$$

A_4^* lattice and its S_5 point group symmetry

- Need five links symmetrically spanning four dimensions $\longrightarrow A_4^*$
  4d analog of 2d triangular lattice – non-orthogonal, degenerate
  Obtain from dimensional reduction with symmetric constraint $\sum_a \partial_a = 0$

- **S_5 point group symmetry:** S_5 irreps match those of $\mathrm{SO}(4)_{tw}$
  Related by orthogonal 5×5 matrix P: $(P^T = P^{-1})$

$$\begin{pmatrix} \psi_\mu \\ \\ \overline{\eta} \end{pmatrix} = P_{\{\mu,5\}a} \begin{pmatrix} \psi_a \end{pmatrix} \qquad \begin{pmatrix} \chi_{\mu\nu} \\ \\ \overline{\psi}_\mu \end{pmatrix} = P_{\mu a} P_{\{\nu,5\}b} \begin{pmatrix} \chi_{ab} \end{pmatrix}$$

(Explicit form of P on next page...)

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{3}{\sqrt{12}} & 0 \\ \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & -\frac{4}{\sqrt{20}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} (\widehat{e}_a)_\mu \\ \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

Supersymmetric lattice action

- **Twisted action:** S is manifestly $\mathcal{Q}$-supersymmetric
  $$S = \int d^4x \; \frac{N}{2\lambda} \text{Tr} \left[\mathcal{Q} \left(\chi_{ab}\mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a \mathcal{A}_a - \frac{1}{2}\eta d \right) - \frac{N}{8\lambda}\epsilon_{abcde} \; \chi_{ab}\overline{\mathcal{D}}_c\chi_{de} \right]$$
  $\mathcal{Q}S = 0$ follows from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity $\epsilon_{abcde} \; \overline{\mathcal{D}}_c\overline{\mathcal{F}}_{de} = 0$

- **Expand:** Apply $\mathcal{Q}$ and integrate out auxiliary field d (implicit trace):
  $$S = \int d^4x \; \frac{N}{2\lambda} \left[-\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} + \frac{1}{2}\left(\overline{\mathcal{D}}_a\mathcal{A}_a\right)^2 - \chi_{ab}\mathcal{D}_{[a}\psi_{b]} - \eta\overline{\mathcal{D}}_a\psi_a - \frac{1}{4}\epsilon_{abcde} \; \chi_{ab}\overline{\mathcal{D}}_c\chi_{de} \right]$$

- As for $\mathcal{Q}$ transformations, lattice action just replaces $\mathcal{A}_a \longrightarrow \mathcal{U}_a$
  (also $\int d^4x \longrightarrow \sum_n$ and $\lambda \longrightarrow \lambda_{\text{lat}}$ with factor of $\det P_{\mu\nu} = 1/\sqrt{d+1}$)

Remarkable analytic consequences

- **Exact symmetries:** gauge invariance + $\mathcal{Q}$ + S_5

- Moduli space preserved to all orders in lattice perturbation theory
  $\longrightarrow$ no scalar potential induced by radiative corrections

- β function vanishes at one loop in lattice perturbation theory

- Real-space RG blocking transformations preserve $\mathcal{Q}$ and S_5
  $\Longrightarrow$ One log tuning to recover all symmetries ($\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$) in continuum

- Can present last if time permits...
  First consider numerical Monte Carlo importance sampling,
  $$\langle\mathcal{O}\rangle = \frac{1}{Z}\int[d\mathcal{U}_a][d\overline{\mathcal{U}}_a][d\Psi]\mathcal{O}e^{-S}$$

Numerical complications $\longrightarrow$ improved action

- Exact zero modes and flat directions must be regulated
- **Complexification complication 1:** U(N) gauge invariance
  U(N) = SU(N) $\otimes$ U(1) but U(1) only decouples in continuum
- **Complication 2:** $\mathcal{Q}\,\mathcal{U}_a = \psi_a \Longrightarrow \mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C})$ (links in algebra)
  Need $\mathcal{U}_a = \mathbb{I}_N + a\mathcal{A}_a + \mathcal{O}(a^2)$ for continuum limit,
  $\longrightarrow$ stabilize by soft susy-breaking scalar potential
  $$\delta S = \frac{N}{2\lambda_{\text{lat}}}\mu^2 \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2$$
  Lifts SU(N) flat directions and bosonic zero modes
- **Susy breaking:** Automatically vanishes as $\mu^2 \to 0$
  Monitor $\mathcal{Q}$ Ward identity violations $\langle \mathcal{Q}\mathcal{O}\rangle \neq 0$: action and $\mathcal{Q}\left[\eta\mathcal{U}_a\overline{\mathcal{U}}_a\right]$
- Flat directions in U(1) sector seem especially problematic
  Include all constant U(1) shifts of x-independent fields, even if $S \neq 0$,
  while SU(N) flat dirs restricted to supersymmeric vacua with $S = 0$
- **U(1) sector:** Impose constraint on plaquette determinant to regulate
  Can be implemented as $\mathcal{Q}$-exact moduli space condition:
  $$\eta\overline{\mathcal{D}}_a\mathcal{U}_a \longrightarrow \eta\left(\overline{\mathcal{D}}_a\mathcal{U}_a + G\sum_{a\neq b}\left[\det\mathcal{P}_{ab} - 1\right]\mathbb{I}_N\right)$$
  Modifies e.o.m. for auxiliary field $d = \overline{\mathcal{D}}_a\mathcal{U}_a + 2G\text{Re}\ \sum_{a<b}\left(\det\mathcal{P}_{ab} - 1\right)\mathbb{I}_N$
- **Improved action:** Ward identity violations $\langle \mathcal{Q}\mathcal{O}\rangle \propto (a/L)^2$
  $\longrightarrow$ effective $\mathcal{O}(a)$ improvement

Ongoing numerical investigations

- **Static potential:** From $r \times T$ Wilson loops $W(r,T) \propto \exp\left[-V(r)T\right]$
  Coulombic with perturbative $C = \lambda/(4\pi)$ at weak couplings $\lambda \leq 4/\sqrt{5}$
  Currently carrying out tree-level-improved analysis,
  $$V(r) \longrightarrow V(r_I), \quad \frac{1}{r_I^2} \equiv 4\pi^2 G(r_\mu) = 4\pi^2\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\frac{\exp(ir\cdot k)}{4\sum_{\mu=1}^4\sin^2(k\cdot\widehat{e}_\mu/2)}$$
- **Scaling dimensions:** For Konishi and SUGRA ($20'$) operators
  $\sum_I \text{Tr}\left[\Phi^I(x)\Phi^I(x)\right] \longrightarrow \sum_a \text{Tr}\left[\phi_a(n)\phi_a(n)\right] - \text{vev}$
  Preliminary finite-size scaling and MCRG $\gamma_m \sim 3\lambda/(4\pi^2)$
  Complication: Mixing with SO(4)$_R$-singlet part of $20'$

Long-distance effective action of the lattice theory

- **Question:** Does full quantum lattice theory produce $\mathcal{N}=4$ SYM in continuum limit (flowing to IR $1/L \to 0$)

- Need symmetry-preserving real-space RG blocking transformation $a \longrightarrow a' = 2a$ with possible rescaling of non-compact variables:

$$\mathcal{U}_a'(n) = \xi\mathcal{U}_a(n)\mathcal{U}_a(n+\widehat{a}) \qquad \overline{\mathcal{U}}_a'(n) = \xi\overline{\mathcal{U}}_a(n+\widehat{a})\overline{\mathcal{U}}_a(n)$$

$$d'(n) = d(n) \qquad \eta'(n) = \eta(n)$$

$$\psi_a'(n) = \mathcal{Q}\mathcal{U}_a'(n) = \xi\left[\psi_a(n)\mathcal{U}_a(n+\widehat{a}) + \mathcal{U}_a(n)\psi_a(n+\widehat{a})\right]$$

$$\begin{aligned}\chi_{ab}'(n) = {}& \frac{\xi^2}{2}\left[\overline{\mathcal{U}}_a(n+\widehat{a}+2\widehat{b})\overline{\mathcal{U}}_b(n+\widehat{a}+\widehat{b}) + \overline{\mathcal{U}}_b(n+2\widehat{a}+\widehat{b})\overline{\mathcal{U}}_a(n+\widehat{a}+\widehat{b})\right]\chi_{ab}(n) \\ & + \xi^2\left[\overline{\mathcal{U}}_a(n+\widehat{a}+2\widehat{b})\chi_{ab}(n+\widehat{b})\overline{\mathcal{U}}_b(n) + \overline{\mathcal{U}}_b(n+2\widehat{a}+\widehat{b})\chi_{ab}(n+\widehat{a})\overline{\mathcal{U}}_a(n)\right] \\ & + \frac{\xi^2}{2}\chi_{ab}(n+\widehat{a}+\widehat{b})\left[\overline{\mathcal{U}}_a(n+\widehat{b})\overline{\mathcal{U}}_b(n) + \overline{\mathcal{U}}_b(n+\widehat{a})\overline{\mathcal{U}}_a(n)\right]\end{aligned}$$

- Including $\eta \to \eta + c\mathbb{I}_N$ and U(1) ghost number, symmetries allow only $\mathcal{Q}\text{Tr}\left[\Psi f(\mathcal{U},\overline{\mathcal{U}},d)\right]$, $\mathcal{Q}\left\{\text{Tr}\left[\eta\right]\text{Tr}\left[f(\mathcal{U},\overline{\mathcal{U}},d)\right]\right\}$ and existing $\mathcal{Q}$-closed term

- Most general renormalizable action (coefficients unconstrained by $\mathcal{Q}$):

$$\begin{aligned}S_{\text{eff}} \sim {}& \mathcal{Q}\text{Tr}\left[\alpha_1\chi_{ab}\mathcal{F}_{ab} + \alpha_2\eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{\alpha_3}{2}\eta d\right] - \frac{\alpha_4}{4}\epsilon_{abcde}\text{Tr}\left[\chi_{de}\overline{\mathcal{D}}_c\chi_{ab}\right] \\ & + \beta\mathcal{Q}\left\{\text{Tr}\left[\eta\mathcal{U}_a\overline{\mathcal{U}}_a\right] - \frac{1}{N}\text{Tr}\left[\eta\right]\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right]\right\}\end{aligned}$$

- "β" term lifts moduli space $\Longrightarrow$ perturbatively forbidden (arXiv:1408.7067) (at non-perturbative level, may need to tune $\beta \to 0$)

$$\begin{aligned}S_{\text{eff}} \sim \frac{N\alpha_1}{2\lambda}\Bigg[& -\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} - \chi_{ab}\mathcal{D}_{[a}\psi_{b]} + \frac{\alpha_2}{\alpha_1}d\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{\alpha_2}{\alpha_1}\eta\overline{\mathcal{D}}_a\psi_a - \frac{\alpha_3}{2\alpha_1}d^2 \\ & - \frac{\alpha_4}{4\alpha_1}\epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}_c\chi_{ab}\Bigg]\end{aligned}$$

- Rescale $\chi \to \sqrt{\frac{\alpha_1}{\alpha_4}}\chi$; $\psi \to \sqrt{\frac{\alpha_4}{\alpha_1}}\psi$; $\eta \to \frac{\alpha_1}{\alpha_2}\sqrt{\frac{\alpha_1}{\alpha_4}}\eta$; $d \to \frac{\alpha_1}{\alpha_2}d$ and $\lambda \to \alpha_1\lambda$:

$$S_{\text{eff}} \sim -\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} - \chi_{ab}\mathcal{D}_{[a}\psi_{b]} + d\overline{\mathcal{D}}_a\mathcal{U}_a - \eta\overline{\mathcal{D}}_a\psi_a - \frac{\alpha_1\alpha_3}{2\alpha_2^2}d^2 - \frac{1}{4}\epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}_c\chi_{ab}$$

- Finally e.o.m. for auxiliary field is now $d = \frac{\alpha_2^2}{\alpha_1\alpha_3}\overline{\mathcal{D}}_a\mathcal{U}_a$, so

$$S_{\text{eff}} \sim -\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} - \chi_{ab}\mathcal{D}_{[a}\psi_{b]} + \frac{\alpha_2^2}{\alpha_1\alpha_3}\frac{1}{2}\left(\overline{\mathcal{D}}_a\mathcal{U}_a\right)^2 - \eta\overline{\mathcal{D}}_a\psi_a - \frac{1}{4}\epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}_c\chi_{ab}$$

$\Longrightarrow$ Only one marginal parameter may need to be tuned to recover full continuum $\mathcal{N}=4$ SYM

The other 15 $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

- Define discrete symmetries $\{R_a, R_{ab}\}$, subgroups of continuum $\text{SO}(6)_R$
  Example: Continuum invariance under transformations

$$
\begin{array}{llll}
R_a\eta = 2\psi_a & R_a\psi_a = \frac{1}{2}\eta & R_a\psi_b = -\chi_{ab} & \\
R_a\chi_{ab} = -\psi_b & R_a\chi_{bc} = \frac{1}{2}\epsilon_{bcagh}\chi_{gh} & (b \neq a) & \\
R_a\mathcal{D}_a = \mathcal{D}_a & R_a\overline{\mathcal{D}}_a = \overline{\mathcal{D}}_a & R_a\mathcal{D}_b = \overline{\mathcal{D}}_b & R_a\overline{\mathcal{D}}_b = \mathcal{D}_b
\end{array}
$$

- Inconsistent with lattice geometry, but can define lattice analog

$$
R_a\mathcal{U}_a = \mathcal{U}_a \qquad R_a\overline{\mathcal{U}}_a = \overline{\mathcal{U}}_a \qquad R_a\mathcal{U}_b = \overline{\mathcal{U}}_b^{-1} \qquad R_a\overline{\mathcal{U}}_b = \mathcal{U}_b^{-1}
$$

- Any one of $\{R_a, R_{ab}\}$ would require $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ and $\beta = 0$,
  guaranteeing restoration of all symmetries of continuum $\mathcal{N} = 4$ SYM
- Qualitatively, $\mathcal{Q}_a = R_a\mathcal{Q}$ while $\mathcal{Q}_{ab} = R_{ab}\mathcal{Q}$
  and the individual $\{R_a, R_{ab}\}$ are related by the S_5 point group symmetry
- Can monitor R_a violation in lattice calculation by measuring (normalized)

$$
R_a\mathcal{W}_{ab} - \mathcal{W}_{ab} \sim \mathcal{U}_a(x)\overline{\mathcal{U}}_b^{-1}(x+\widehat{a})\overline{\mathcal{U}}_a(x+\widehat{b})\mathcal{U}_b^{-1}(x) - \mathcal{U}_a(x)\mathcal{U}_b(x+\widehat{a})\overline{\mathcal{U}}_a(x+\widehat{b})\overline{\mathcal{U}}_b(x)
$$

- Tune parameter $\frac{\alpha_2^2}{\alpha_1\alpha_3}$ in S_{eff} above to minimize R_a violation
  when approaching long-distance continuum limit

Sign problem

- **Phase reweighting:** Can enable importance sampling Monte Carlo
  using real non-negative Boltzmann factor $|\text{pf}\,\mathcal{D}|e^{-S_B}$

$$
\begin{aligned}
\langle\mathcal{O}\rangle &= \frac{1}{Z}\int[d\mathcal{U}_a][d\overline{\mathcal{U}}_a][d\Psi]\mathcal{O}e^{-S_B[\mathcal{U}_a,\overline{\mathcal{U}}_a]-\Psi^T\mathcal{D}[\mathcal{U}_a,\overline{\mathcal{U}}_a]\Psi} \\
&= \frac{1}{Z}\int[d\mathcal{U}_a][d\overline{\mathcal{U}}_a]\mathcal{O}e^{i\alpha}|\text{pf}\,\mathcal{D}|e^{-S_B} = \frac{\left\langle\mathcal{O}e^{i\alpha}\right\rangle_{pq}}{\left\langle e^{i\alpha}\right\rangle_{pq}}
\end{aligned}
$$

- **Sign problem:** When the phase α fluctuates
  so much that $\left\langle e^{i\alpha}\right\rangle_{pq}$ is consistent with zero
- **Numerical results:** Phase fluctuations become significant for $\lambda_{\text{lat}} \gtrsim 5$;
  appear largely independent of volume (unlike finite-density QCD)
- **Strange behavior:** Phase $e^{i\alpha}$ is extremely sensitive to temporal BCs
  $e^{i\alpha} \approx 1$ with anti-periodic BCs, $\left\langle e^{i\alpha}\right\rangle_{pq} \approx 0$ with periodic BCs
  Even more strangely, other observables change little for different BCs