Phases of a strongly coupled four-fermion theory

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arXiv:1609.08541 and work in progress
with Simon Catterall & Nouman Butt

Overview: Motivation, system, phase diagram results

Topological insulators $\longrightarrow$ phase transitions with no local order param.

Possibility of novel continuum limit at strong coupling?

Simple 4d system with four **reduced staggered** fermions (no gauge)

SU(4) flavor symmetry and shift symmetries forbid bilinear mass terms:

$$S = \eta_\mu \psi^a \Delta_\mu \psi^a - \frac{1}{4} G^2 \, \epsilon_{abcd} \, \psi^a \psi^b \psi^c \psi^d$$

Question: Do symmetries break spontaneously?

So far we find **no** symmetry-breaking bilinear condensates for any G

For large G, fermion mass from symmetric four-fermion condensate

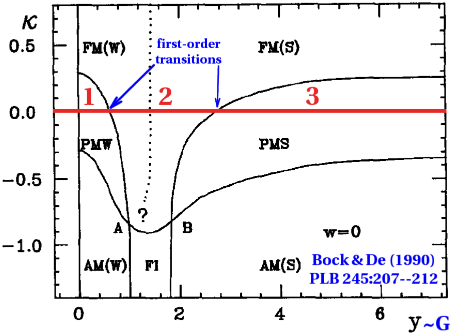
Symmetric massless and massive phases separated by narrow region exhibiting long-range correlations & apparently continuous transition(s)

Standard picture of Higgs–Yukawa phase diagram

Auxiliary field rep. with self-dual scalar σ^+_{ab} $\longrightarrow$ positive pfaffian

$$S = \eta_\mu \psi^a \Delta_\mu \psi^a + G \psi^a \sigma^+_{ab} \psi^b + \frac{1}{4}\left(\sigma^+_{ab}\right)^2$$

Similar to $\kappa = 0$ line of Higgs–Yukawa systems studied in past
(with different fermion discretizations and different lattice symmetries)



In past studies, the symmetric massless (1) and massive (3) phases are separated by broad broken phase (2) with bilinear condensates

Massive symmetric strong-coupling limit

The strong-coupling limit behaves the same as in past studies

(Eichten & Preskill, 1986)

Strong-coupling propagators for ψ and composite boson $B \simeq \psi\psi$

$$F(p) = \frac{i\sqrt{6G^2}\sum_\mu \sin p_\mu}{\sum_\mu \sin^2 p_\mu + m_F^2} \qquad B(p) = \frac{8\left(6G^2\right)}{4\sum_\mu \sin^2(p_\mu/2) + m_B^2}$$

$$m_F^2 = 4\left(6G^2\right)^2 - 2 \qquad m_B^2 = 4\left(6G^2\right) - 8$$

$\left\langle \epsilon_{abcd}\, \psi^a\psi^b\psi^c\psi^d \right\rangle \longrightarrow$ non-zero masses without symmetry breaking

Can interpret condensate as $\langle \psi^a \Psi^a \rangle$ with $\Psi^a \equiv \epsilon_{abcd}\, \psi^b\psi^c\psi^d$

Absence of bilinear condensates for all G

Two analytic arguments rely on reduced staggered lattice symmetries,
don't hold for systems studied in past

1. **Analog of Vafa–Witten theorem**

   Add hermitian, symmetry-breaking $\delta S = im\,\psi^a\left(i\sigma_2 \otimes \mathbb{I}\right)_{ab}\psi^b$

   For small m, Schwarz inequality $Z(m) \leq Z(0)$

   $$\Longrightarrow \left\langle \psi^a\left(i\sigma_2 \otimes \mathbb{I}\right)_{ab}\psi^b \right\rangle = \left.\frac{\delta \log Z}{\delta m}\right|_{m=0} = 0 \text{ independent of } G$$

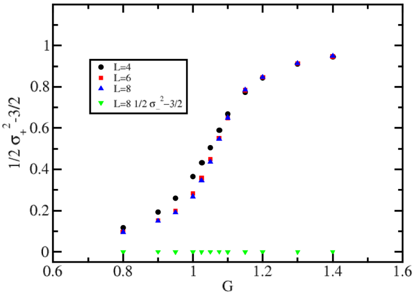
2. **One-loop Coleman–Weinberg effective potential**

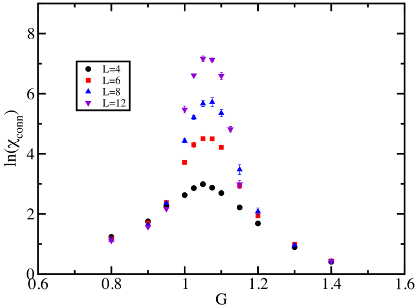
   Integrating over fermions for constant $\sigma^+_{ab} = \mu\left(i\sigma_2 \otimes \mathbb{I}\right)_{ab}$,

   $$S_{\text{eff}} = -\frac{1}{2}\text{Tr}\left[\log\left(\eta_\mu \Delta_\mu + G\sigma^+_{ab}\right)\right] \Longrightarrow V_{\text{eff}} = -\sum_k \log\left|\lambda_k^2 - G^2\mu^2\right|$$

   Local minimum for symmetric $\sigma^+_{ab} = 0 \Longrightarrow \left\langle \psi^a\psi^b \right\rangle = 0$ for all G

Numerical results for phase diagram





$\langle \sigma_+^2 \rangle \sim \langle \psi\psi\psi\psi \rangle$ moves smoothly from weak- to strong-coupling limit

No visible finite-volume effects for $L \geq 6$

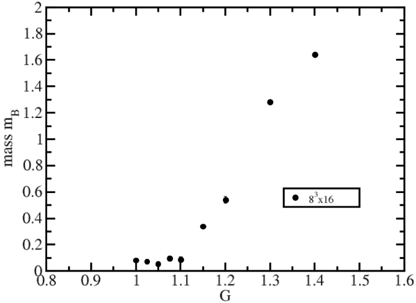
Connected susceptibility $\chi_{\text{conn}} = \frac{1}{V} \sum_{x,y,a,b} \left\langle \psi^a(x)\psi^b(x)\psi^a(y)\psi^b(y) \right\rangle$

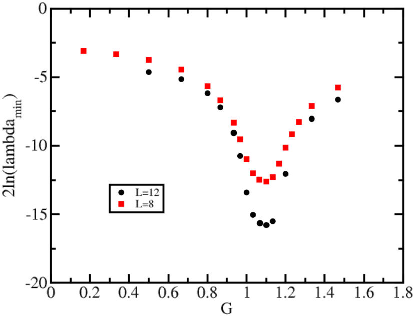
scales $\chi_{\text{conn}} \sim L^4$ in transition region $G \approx 1.05$

Disconnected susceptibility $\chi_{\text{dis}} \sim \left[\sum \left\langle \psi^a\psi^b \right\rangle\right]^2 = 0$

Long-range correlations in transition region

In addition to $\chi_{\text{conn}} \sim L^d$, two more signals of long-range correlations in transition region $G \approx 1.05$





Mass of composite boson $B \simeq \psi\psi$ very small throughout $1 \lesssim G \lesssim 1.1$, before growing $m_B \sim G$ as in strong-coupling expansion

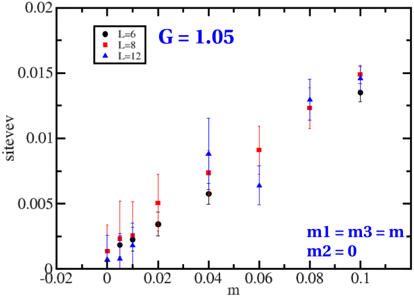
Critical slowing down from rapid drop in smallest eigenvalues of fermion operator (limiting us to $L \leq 12$ so far)

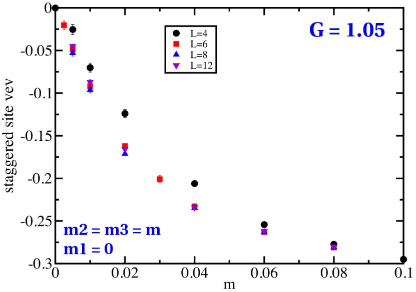
No spontaneous symmetry breaking in transition region

Add three explicit symmetry-breaking sources,
extrapolate each to zero in $L \to \infty$ thermodynamic limit

$$m_1 \; [\psi\psi]^+_{ab}(x) \;\; [i\sigma_2 \otimes \mathbb{I}]_{ab} \qquad 2[\psi\psi]^+_{ab} = \psi^a\psi^b + \epsilon_{abcd}\psi^c\psi^d / 2$$

$$m_2 \; \epsilon(x) \, [\psi\psi]^+_{ab}(x) \;\; [i\sigma_2 \otimes \mathbb{I}]_{ab} \qquad m_3 \; \epsilon(x)\xi_\mu(x)\psi^a(x)\psi^a(x \pm \widehat{\mu})$$





As above, no visible finite-volume effects for $L \geq 6$,
and all condensates vanish as sources removed $\Longrightarrow$ no SSB

Much larger response to “staggered” mass term including $\epsilon(x)$

Recapitulation

Symmetric massless and massive phases
in simple 4d **reduced staggered** four-fermion theory

Long-range correlations in narrow intermediate region
with apparently continuous transition(s)

Analytic arguments and numerical results
show no symmetry-breaking bilinear condensates for any G

Larger lattice volumes may require removing critical slowing down
(e.g., multigrid RHMC)

Currently exploring use of strong gauge coupling
to induce four-fermion interaction

Thank you!

Collaborators

Simon Catterall, Nouman Butt

Funding and computing resources

Backup: Positivity of pfaffian

$$S = \eta_\mu \psi^a \Delta_\mu \psi^a + G \psi^a \sigma^+_{ab} \psi^b + \frac{1}{4} \left(\sigma^+_{ab}\right)^2$$

Self-dual scalar $\sigma^+_{ab} = \frac{1}{2}\left(\sigma_{ab} + \frac{1}{2}\epsilon_{abcd}\,\sigma_{cd}\right)$

transforms in $(3, 1)$ rep of SU(2)$\times$SU(2) $\subset$ SU(4)

Fermion operator $M = \eta_\mu \Delta_\mu + G\sigma^+_{ab}$ is real and anti-symmetric

$\Longrightarrow$ paired purely imaginary eigenvalues $\pm i\lambda$

Invariance under one SU(2) subgroup $\Longrightarrow$ each of $\pm i\lambda$ is doubled

Exact λ^2 quartets checked numerically,

guarantee real positive pfaffian (i.e., no sign problem)

Backup: More about bilinear condensates

The Vafa–Witten-style argument relies on the positivity of the pfaffian

For $S = S_0 + im\,\psi^a \left(i\sigma_2 \otimes \mathbb{I}\right)_{ab} \psi^b$ with small m we have

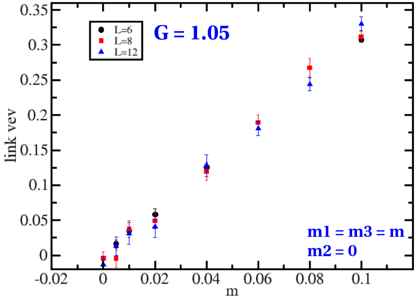
$$
\begin{aligned}
Z(m) &= \int D\sigma^+ D\psi \, e^{-S_0} \left(1 + im \left(i\sigma_2 \otimes \mathbb{I}\right)_{ab} \psi^a \psi^b + \mathcal{O}\left(m^2\right)\right) \\
&= \int D\sigma^+ \, \mathrm{pf}\left(M\right) e^{\sigma_+^2/2} \left(1 + im \left(i\sigma_2 \otimes \mathbb{I}\right)_{ab} M^{-1}_{ab} + \mathcal{O}\left(m^2\right)\right) \\
&\simeq \int D\sigma^+ \, \mathrm{pf}\left(M\right) e^{\sigma_+^2/2} \exp\left[im \left(i\sigma_2 \otimes \mathbb{I}\right)_{ab} M^{-1}_{ab}\right] \le Z(0)
\end{aligned}
$$

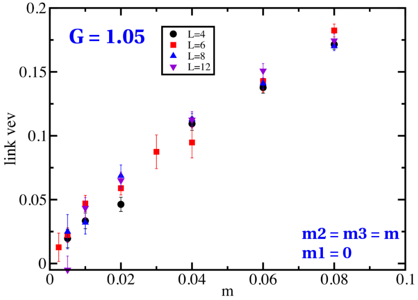
Since $\mathrm{pf}\left(M\right) e^{\sigma_+^2/2}$ is positive and $m \left(i\sigma_2 \otimes \mathbb{I}\right)_{ab} M^{-1}_{ab}$ is real,
integral trivially maximized by vanishing phase $e^{im} = 1$

Argument also holds for staggered and one-link bilinears,
but not for fermion discretizations that allow hermitian $\overline{\psi}\psi$ bilinears

Backup: One-link condensate in transition region

$$m_3\ \epsilon(x)\xi_\mu(x)\psi^a(x)\left[\psi^a(x+\widehat{\mu})+\psi^a(x-\widehat{\mu})\right]\quad \text{with}\quad \xi_\mu(x)=(-1)^{\sum_{i=\mu+1}^{d-1}x_i}$$



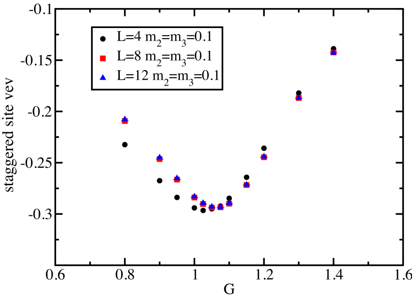


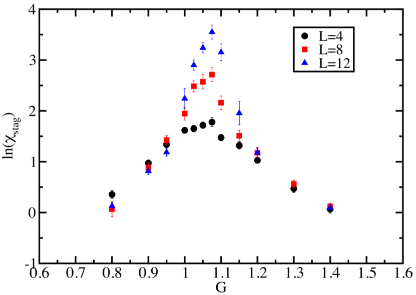
As for other condensates, no visible finite-volume effects for $L \geq 6$

As for other condensates, vanishes as sources removed $\Longrightarrow$ no SSB

Backup: Staggered site susceptibility

$$m_2\, \epsilon(x)\psi^a(x)\,[i\sigma_2 \otimes \mathbb{I}]_{ab}\,\psi^b(x) \qquad \text{with} \qquad \epsilon(x) = (-1)^{\sum_{i=0}^{d-1} x_i}$$





System exhibits larger response to non-zero m_2
compared to non-staggered m_1 of same magnitude

Staggered susceptibility grows much more slowly with volume,
$\chi_{\text{conn}}^{\text{stag}} \sim L^{3/2}$ rather than $\chi_{\text{conn}} \sim L^4$