$\mathcal{N} = 4$ supersymmetric Yang–Mills on a space-time lattice

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arXiv:1405.0644, arXiv:1410.6971, arXiv:1411.0166 & more to come
with Simon Catterall, Poul Damgaard, Tom DeGrand and Joel Giedt

Plan

- Motivations for lattice supersymmetry in general
- Why lattice $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) in particular
- Some highlights of the lattice $\mathcal{N} = 4$ SYM formulation
  & demonstrations of correctness
- Some initial physics results
  & connections to perturbation theory, AdS/CFT, bootstrap
- The future

Context: Why lattice supersymmetry

Lattice discretization provides non-perturbative,
gauge-invariant regularization of vectorlike gauge theories

Amenable to numerical analysis
$\longrightarrow$ complementary approach to study strongly coupled field theories

Proven success for QCD; many potential susy applications:

- Compute Wilson loops, spectrum, scaling dimensions, etc., complementing perturbation theory, holography, bootstrap, ...
- Further direct checks of conjectured dualities
- Predict low-energy constants from dynamical susy breaking
- Validate or refine AdS/CFT-based modelling (e.g., QCD phase diagram, condensed matter systems)

Context: Why not lattice supersymmetry

There is a problem with supersymmetry on a space-time lattice

Recall: supersymmetry extends Poincaré symmetry
by spinorial generators Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ with $\mathrm{I} = 1, \cdots, \mathcal{N}$

The resulting algebra includes $\left\{Q_{\alpha}, \overline{Q}_{\dot{\alpha}}\right\} = 2\sigma^{\mu}_{\alpha\dot{\alpha}} P_{\mu}$

P_{μ} generates infinitesimal translations, which don't exist on the lattice
$\Longrightarrow$ supersymmetry explicitly broken at classical level

Consequence for lattice calculations

Quantum effects generate (typically many) susy-violating operators

Fine-tuning their couplings to restore susy is generally not practical

Why $\mathcal{N} = 4$ SYM: Exact susy on the lattice

In order to forbid generation of susy-violating operators
(some subset of) the susy algebra must be preserved

In four dimensions $\mathcal{N} = 4$ SYM is the only known system
with a supersymmetric lattice formulation

$\mathcal{N} = 4$ SYM is an extremely interesting theory

- SU(*N*) gauge theory with four fermions Ψ^{I} and six scalars Φ^{IJ},
all massless and in adjoint rep.
- Action consists of kinetic, Yukawa and four-scalar terms
- Supersymmetric: 16 supercharges Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ with $\mathrm{I} = 1, \cdots, 4$
Fields and *Q*'s transform under global SU(4) $\simeq$ SO(6) R symmetry
- Conformal: β function is zero for all 't Hooft couplings λ

Why lattice $\mathcal{N} = 4$ SYM: Kähler–Dirac fermions

What is special about $\mathcal{N} = 4$ SYM

The 16 fermionic supercharges Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ of $\mathcal{N} = 4$ SYM fill a Kähler–Dirac multiplet:

$$Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{I}}_{\dot{\alpha}} \longrightarrow \mathcal{Q}, \mathcal{Q}_{\mu}, \mathcal{Q}_{\mu\nu}, \mathcal{Q}_{\mu\nu\rho}, \mathcal{Q}_{\mu\nu\rho\sigma}$$

all totally anti-symmetric

In this notation there is a susy subalgebra $\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$

This can be exactly preserved on the lattice

Let's see how…

Dijkstra, “Notes on Structured Programming”, 1970

A convincing demonstration of correctness being impossible
as long as the mechanism is regarded as a black box,
our only hope lies in not regarding the mechanism as a black box.

Kähler–Dirac fermions from topological twisting

The Kähler–Dirac representation is related to the usual $Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ by

$$\begin{pmatrix} Q^1_{\alpha} & Q^2_{\alpha} & Q^3_{\alpha} & Q^4_{\alpha} \\ \overline{Q}^1_{\dot{\alpha}} & \overline{Q}^2_{\dot{\alpha}} & \overline{Q}^3_{\dot{\alpha}} & \overline{Q}^4_{\dot{\alpha}} \end{pmatrix} = \mathcal{Q} + \gamma_{\mu}\mathcal{Q}_{\mu} + \gamma_{\mu}\gamma_{\nu}\mathcal{Q}_{\mu\nu} + \gamma_{\mu}\gamma_5\mathcal{Q}_{\mu\nu\rho} + \gamma_5\mathcal{Q}_{\mu\nu\rho\sigma}$$

$$\longrightarrow \mathcal{Q} + \gamma_a\mathcal{Q}_a + \gamma_a\gamma_b\mathcal{Q}_{ab}$$

with $a, b = 1, \cdots, 5$

The 4×4 matrix involves R symmetry transformations along each row and (euclidean) Lorentz transformations along each column

$\Longrightarrow$ Kähler–Dirac components transform under "twisted rotation group"

$$\mathrm{SO}(4)_{tw} \equiv \mathrm{diag}\left[\mathrm{SO}(4)_{\mathrm{euc}} \otimes \underset{\uparrow}{\mathrm{SO}(4)_R}\right]$$

only $\mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$

Twisted $\mathcal{N} = 4$ SYM

$$\text{SO}(4)_{tw} \equiv \text{diag}\Big[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R\Big]$$

- $\mathcal{Q}$, $\mathcal{Q}_\mu$, $\mathcal{Q}_{\mu\nu}$, …transform with **integer spin** – no longer spinors!
- Fermions decompose in the same way, $\Psi^I \longrightarrow \{\eta, \psi_a, \chi_{ab}\}$
- Scalar fields transform as a four-vector B_μ plus two scalars ϕ, $\overline{\phi}$
  Combine with A_μ in complexified five-component gauge field

$$\mathcal{A}_a = A_a + iB_a = (A_\mu, \phi) + i(B_\mu, \overline{\phi}) \qquad \text{and similarly for } \overline{\mathcal{A}}_a$$

Complexified gauge field $\Longrightarrow$ U(N) = SU(N) $\otimes$ U(1) gauge invariance

Irrelevant in the continuum, but will affect lattice calculations

Twisted $\mathcal{N} = 4$ SYM

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Twisting is trustworthy

In flat space just a change of variables, no effect on physics

If you're not yet convinced...

You can derive precisely the same lattice system
from orbifolding / dimensional deconstruction – cf. arXiv:0903.4881

Five-component notation lets us move to the lattice

Goal: Preserve $\mathcal{Q}$ supersymmetry on the lattice

1. $\mathcal{Q}^2 \cdot = 0$
2. $\mathcal{Q}$ directly interchanges bosonic $\longleftrightarrow$ fermionic d.o.f.

Both conditions are easy to verify in five-component notation:

$$\begin{aligned}
\mathcal{Q}\,\mathcal{U}_a &= \psi_a & \qquad\qquad \mathcal{Q}\,\psi_a &= 0 \\
\mathcal{Q}\,\chi_{ab} &= -\overline{\mathcal{F}}_{ab} & \qquad\qquad \mathcal{Q}\,\overline{\mathcal{U}}_a &= 0 \\
\mathcal{Q}\,\eta &= d & \qquad\qquad \mathcal{Q}\,d &= 0
\end{aligned}$$

- Gauge field $\mathcal{U}_a$ and ψ_a live on links between lattice sites
  $\mathcal{U}_a$ must be elements of algebra $\mathfrak{gl}(N, \mathbb{C})$
  $\Longrightarrow$ Non-trivial to ensure $\mathcal{U}_a \longrightarrow \mathbb{I} + \mathcal{A}_a$ in the continuum limit
- Field strength $\overline{\mathcal{F}}_{ab}$ and χ_{ab} live on diagonals of oriented faces
- Bosonic auxiliary field d and η live on sites
  Usual equation of motion: $d = \overline{\mathcal{D}}_a\,\mathcal{U}_a$

Five links in four dimensions: The A_4^* lattice

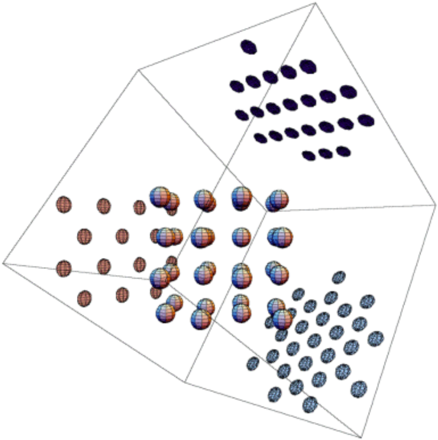
$A_a = (A_\mu, \phi)$ may remind you of dimensional reduction

On the lattice we need to treat all five $\mathcal{U}_a$ symmetrically

—Start with hypercubic lattice
in 5d momentum space

—**Symmetric** constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

—Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in real space

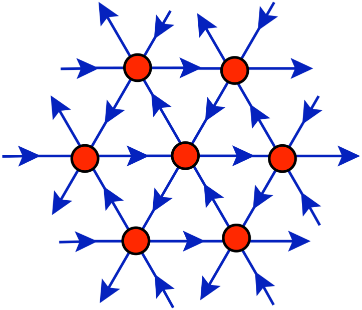


Twisted SO(4) symmetry on the A_4^* lattice

—Can picture A_4^* lattice
as 4d analog of 2d triangular lattice

—Five basis vectors are non-orthogonal
and linearly dependent

—Preserves S_5 point group symmetry



S_5 irreps precisely match onto irreps of twisted $\text{SO}(4)_{tw}$

$$\mathbf{5} = \mathbf{4} \oplus \mathbf{1}: \quad \mathcal{U}_a \longrightarrow \mathcal{A}_\mu, \ \ \phi$$
$$\psi_a \longrightarrow \psi_\mu, \ \ \overline{\eta}$$
$$\mathbf{10} = \mathbf{6} \oplus \mathbf{4}: \quad \chi_{ab} \longrightarrow \chi_{\mu\nu}, \ \ \overline{\psi}_\mu$$

Checkpoint: What we have so far

Thanks to twisting & A_4^*

we have a manifestly supersymmetric lattice action for $\mathcal{N} = 4$ SYM

$$S = \frac{N}{2\lambda_{\text{lat}}} \mathcal{Q} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a \mathcal{U}_a - \frac{1}{2} \eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde}\ \chi_{ab} \overline{\mathcal{D}}_c \chi_{de}$$

$\mathcal{Q}S = 0$ follows from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity

- We have exact U(N) gauge invariance
- We exactly preserve $\mathcal{Q}$, one of 16 supersymmetries
- The S_5 point group symmetry
  provides twisted R & Lorentz symmetry in the continuum limit

Checkpoint: What we have so far

$$S = \frac{N}{2\lambda_{\text{lat}}} \mathcal{Q} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a \mathcal{U}_a - \frac{1}{2} \eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde}\ \chi_{ab} \overline{\mathcal{D}}_c \chi_{de}$$

The high degree of symmetry has important consequences

- Moduli space preserved to all orders of lattice perturbation theory $\longrightarrow$ no scalar potential induced by radiative corrections
- β function vanishes at one loop (at least)
- Real-space RG blocking transformations preserve $\mathcal{Q}$ & S_5
- Only one marginal tuning to recover $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ in the continuum

The theory is **almost** suitable for practical numerical calculations. . .

Stabilizing numerical calculations

We need to add two deformations to the $\mathcal{Q}$-invariant action

Both deal with features required by the supersymmetric construction

Scalar potential to regulate flat directions

$\mathcal{U}_a$ in algebra $\longrightarrow$ Add scalar potential $\left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2$
to ensure $\mathcal{U}_a \longrightarrow \mathbb{I} + \mathcal{A}_a$ in the continuum limit

Otherwise $\mathcal{U}_a$ can run away along flat directions

Plaquette determinant to suppress U(1) sector of U(N)

$\mathcal{U}_a$ complexified $\longrightarrow$ Add approximate SU(N) projection $\left|\det \mathcal{P}_{ab} - 1\right|^2$
where $\mathcal{P}_{ab}$ is the product of four $\mathcal{U}_a$ around the elementary plaquette

Otherwise encounter strong-coupling U(1) confinement transition

Lattice action for $\mathcal{N} = 4$ numerical calculations

Applying $\mathcal{Q}$, integrating out *d* and adding the deformations, we have

$$S = \frac{N}{2\lambda_{\text{lat}}} \left[-\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab} + \frac{1}{2}\left(\overline{\mathcal{D}}_a\mathcal{U}_a\right)^2 - \chi_{ab}\mathcal{D}_{[a}\psi_{b]} - \eta\overline{\mathcal{D}}_a\psi_a \right]$$
$$- \frac{N}{8\lambda_{\text{lat}}}\epsilon_{abcde}\ \chi_{de}\overline{\mathcal{D}}_c\chi_{ab}$$
$$+ \frac{N}{2\lambda_{\text{lat}}}\mu^2\left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2 + \kappa\left|\det\mathcal{P}_{ab} - 1\right|^2$$

Deformations introduce soft supersymmetry breaking

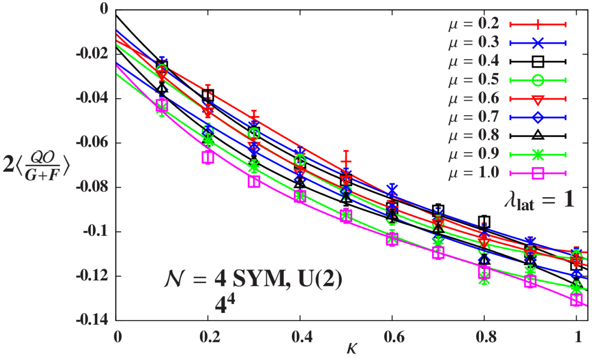
As written, both the μ^2 and κ terms softly break $\mathcal{Q}$

New development: Either (maybe both) can be made $\mathcal{Q}$-invariant by modifying the equation of motion for *d*

Soft $\mathcal{Q}$ supersymmetry breaking

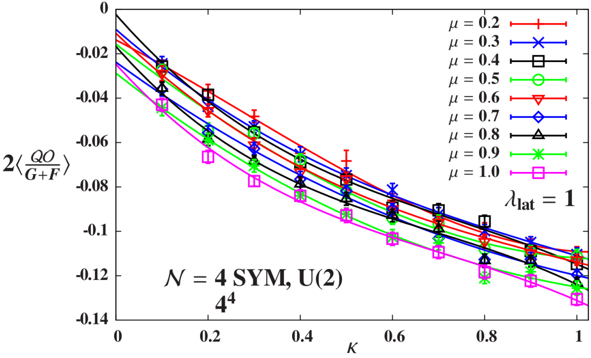
Exact preservation of $\mathcal{Q}$ by the underlying formulation
$\Longrightarrow$ all susy-violating operators automatically vanish as $\mu, \kappa \longrightarrow 0$

Use violations of (normalized) Ward identities $\langle \mathcal{Q}\mathcal{O} \rangle = 0$
to monitor $\mathcal{Q}$ breaking and restoration



Soft $\mathcal{Q}$ supersymmetry breaking

Use normalized Ward identity violations to monitor $\mathcal{Q}$ restoration

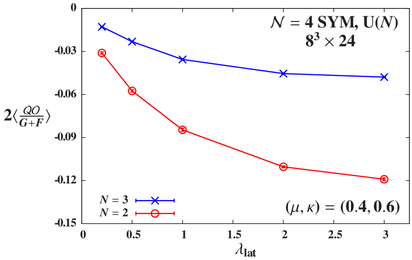


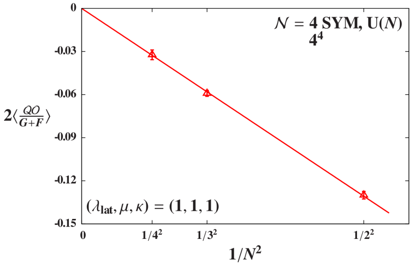
Relatively narrow band

$\Longrightarrow$ more severe $\mathcal{Q}$ breaking from plaquette determinant deformation than from scalar potential deformation

Soft $\mathcal{Q}$ supersymmetry breaking

Use normalized Ward identity violations to monitor $\mathcal{Q}$ restoration





Two more sanity tests with fixed non-zero μ and κ

- Ward identity violations vanish in free-field limit $\lambda_{\text{lat}} \to 0$
- Ward identity violations suppressed $\propto 1/N^2$ as N increases

Effects of soft $\mathcal{Q}$-breaking are under control, at the 1–10% level

Numerical calculations are practical

Recap: Highlights of the formulation

- Topological twisting puts fields in convenient form
- A_4^* lattice provides corresponding discretization
- Soft $\mathcal{Q}$-breaking deformations allow practical numerical studies

The construction is obviously very complicated

To reduce this barrier to entry,
we make our efficient parallel code publicly available

github.com/daschaich/susy

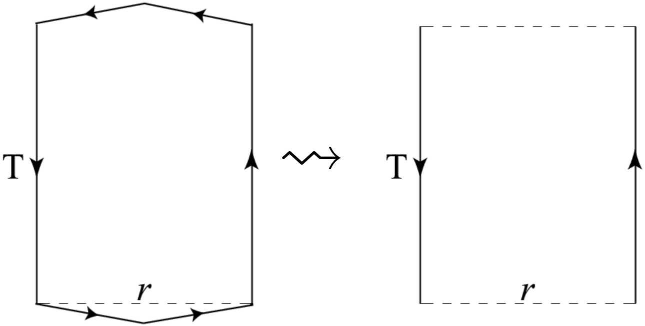
Now for some physics results

- Static potential from Wilson loops
- Konishi operator scaling dimension from Monte Carlo RG

$\mathcal{N} = 4$ static potential from Wilson loops

Extract static potential $V(r)$
from $r \times T$ Wilson loops: $W(r, T) \propto e^{-V(r)\,T}$

Coulomb gauge trick from lattice QCD reduces A_4^* lattice complications



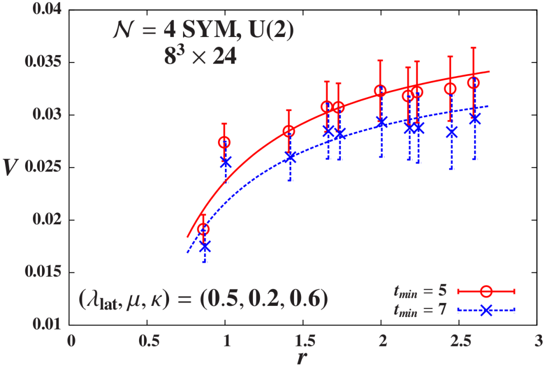
Potential should be Coulombic at all couplings

Current results for $V(r)$ from $W(r, T) \propto e^{-V(r)\,T}$ are fairly noisy

Fit $V(r)$ to Coulombic
or confining form

$$V(r) = A - C/r$$

$$V(r) = A - C/r + \sigma r$$



Always find vanishing string tension $\sigma = 0$
$\longrightarrow$ $V(r)$ is Coulombic for all λ, as expected

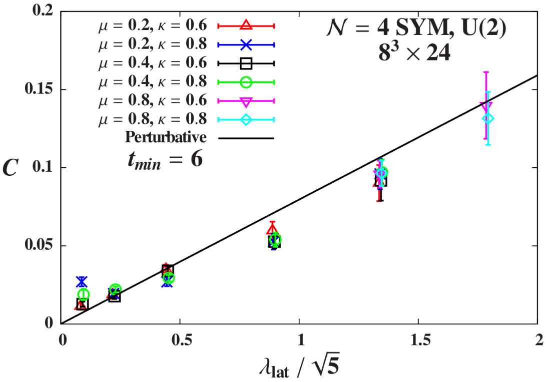
Coupling dependence of $\mathcal{N} = 4$ static potential

We have a Coulombic potential $V(r) = A - C/r$ for all λ

Perturbation theory predicts

$$C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$$

Although noisy, for U(2) our results agree for $\lambda \lesssim 2$
(The $\sqrt{5}$ comes from A_4^*)



No dependence on μ or $\kappa \longrightarrow$ apparently insensitive to soft $\mathcal{Q}$ breaking

Coulomb coefficients for larger U(N)

For $N = 3$ we still have Coulombic $V(r) = A - C/r$

Our results now deviate from (well-convergent) perturbation theory

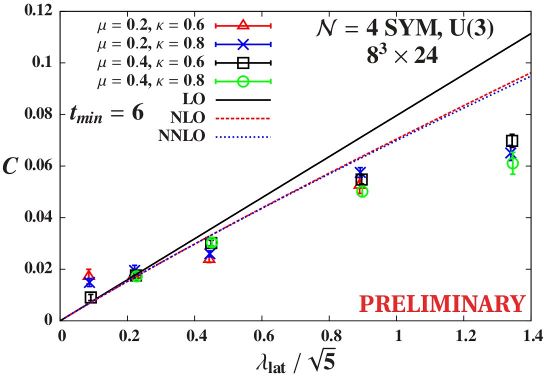
Are we seeing onset of expected large-N strong-coupling behavior?

AdS/CFT predicts

$$C(\lambda) \propto \sqrt{\lambda}$$

for $N \to \infty$, $\lambda \to \infty$, $\lambda \ll N$

In QCD, SU(3) is 'large N'
for some quantities...



Konishi operator scaling dimension

Recall $\mathcal{N} = 4$ SYM is conformal
$\Longrightarrow$ All correlation functions decay algebraically $\propto r^{-\Delta}$

The Konishi operator is the simplest conformal primary operator

$$\mathcal{O}_K = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}\Phi^{\mathrm{I}}\right] \qquad C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) = Ar^{-2\Delta_K}$$

There are many predictions for the scaling dim. $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$

- From perturbation theory for small λ,
  related to $\lambda \to \infty$ by S duality under $\frac{4\pi N}{\lambda} \longleftrightarrow \frac{\lambda}{4\pi N}$
- From holography for $N \to \infty$ and $\lambda \to \infty$ but $\lambda \ll N$
- Bounds on max $\{\Delta_K\}$ from the conformal bootstrap program

We will add lattice gauge theory to this list

Konishi operator on the lattice

$$\mathcal{O}_K = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}\Phi^{\mathrm{I}}\right]$$

On the lattice the scalars Φ^{I} are twisted
and wrapped up in the complexified gauge field $\mathcal{U}_a$

Since $\mathcal{U}_a \approx \mathbb{I} + \mathcal{A}_a$ the most obvious way to extract the scalars is

$$\widehat{\varphi}^a = \mathcal{U}_a\overline{\mathcal{U}}_a - \frac{1}{N}\mathrm{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right]\mathbb{I}$$

This is still twisted, so all $\{a, b\}$ contribute to R-singlet Konishi

$$\widehat{\mathcal{O}}_K = \sum_{a,\ b} \mathrm{Tr}\left[\widehat{\varphi}^a\widehat{\varphi}^b\right]$$

Small-volume lattice Konishi sanity test

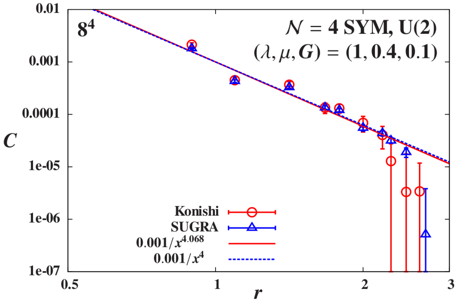
$$\mathcal{O}_K = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}\Phi^{\mathrm{I}}\right] \longrightarrow \widehat{\mathcal{O}}_K = \sum_{a,\ b} \mathrm{Tr}\left[\widehat{\varphi}^a\widehat{\varphi}^b\right]$$

$$\widehat{C}_K(r) \equiv \widehat{\mathcal{O}}_K(x+r)\widehat{\mathcal{O}}_K(x) \propto r^{-2\Delta_K}$$

Qualitative agreement with power laws using perturbative Δ

"SUGRA" operator is R-nonsinglet with protected $\Delta_S = 2$

Need $\mathcal{Q}$-invariant SU(N) projection for test to pass on 8^4 lattice volume



Obviously not a stable way to determine Δ_K — we have other tools

Scaling dimensions from Monte Carlo RG

Couplings flow under RG blocking transformation R_b

n-times-blocked system is $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Consider linear expansion around fixed point $H^\star$ with couplings $c_i^\star$

$$c_i^{(n)} - c_i^\star = \sum_j \left.\frac{\partial c_i^{(n)}}{\partial c_j^{(n-1)}}\right|_{H^\star} \left(c_j^{(n-1)} - c_j^\star\right) \equiv \sum_j T_{ij}^\star \left(c_j^{(n-1)} - c_j^\star\right)$$

$T_{ij}^\star$ is the "stability matrix"

Eigenvalues of $T_{ij}^\star$ are scaling dimensions of corresponding operators

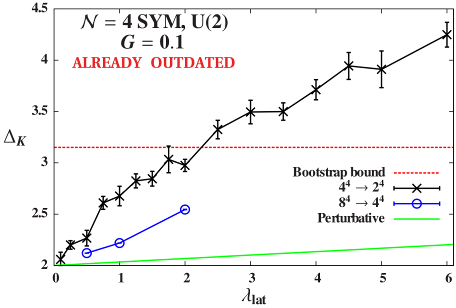
Initial Konishi Δ_K from Monte Carlo RG

Eigenvalues of $T^\star_{ij}$ are scaling dimensions of corresponding operators

Simplest possible trial:
One operator ($\widehat{\mathcal{O}}_K$)
One blocking $n = 1$

Correctly find $\Delta_K \to 2$ as $\lambda \to 0$

Significant volume dependence
$\longrightarrow$ approach perturbation theory as L increases



Many refinements (and other approaches) currently in the works

Recapitulation

- Lattice gauge theory provides a complementary approach to study strongly coupled supersymmetric field theories
- Lattice $\mathcal{N} = 4$ SYM based on topological twisting exactly preserves subset of susy algebra, $\mathcal{Q}^2 = 0$
  Allows practial numerical calculations
- The construction is complicated $\longrightarrow$ publicly-available code to reduce barriers to entry
- The static potential is always Coulombic
  For $N = 2$ $C(\lambda)$ is consistent with perturbation theory
  For $N = 3$ an intriguing discrepancy at stronger couplings
- Work is progressing to predict the Konishi scaling dimension
- There are many more directions to pursue in the future

One future direction: Understanding the sign problem

In lattice gauge theory we compute operator expectation values

$$\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}}\int[d\mathcal{U}][d\overline{\mathcal{U}}]\,\mathcal{O}\,e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]}\,\text{pf}\,\mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

$\text{pf}\,\mathcal{D} = |\text{pf}\,\mathcal{D}|e^{i\alpha}$ is generically complex for lattice $\mathcal{N} = 4$ SYM
$\longrightarrow$ Complicates interpretation of $\left[e^{-S_B}\,\text{pf}\,\mathcal{D}\right]$ as Boltzmann weight

Still possible to “reweight” “phase-quenched (pq)” calculations

$$\langle\mathcal{O}\rangle_{pq} = \frac{1}{\mathcal{Z}_{pq}}\int[d\mathcal{U}][d\overline{\mathcal{U}}]\,\mathcal{O}\,e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]}\,|\text{pf}\,\mathcal{D}| \qquad \langle\mathcal{O}\rangle = \frac{\left\langle\mathcal{O}e^{i\alpha}\right\rangle_{pq}}{\left\langle e^{i\alpha}\right\rangle_{pq}}$$

Sign problem: This breaks down if $\left\langle e^{i\alpha}\right\rangle_{pq}$ consistent with zero

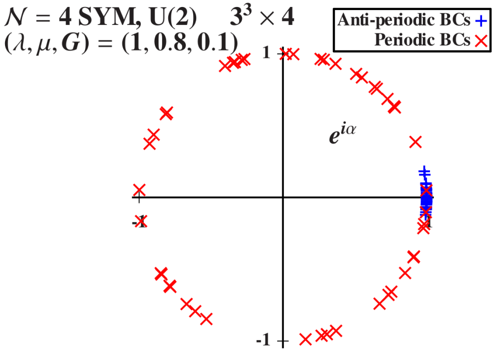
Illustration of sign problem and its absence

- With periodic temporal fermion boundary conditions
  we have an obvious sign problem, $\left\langle e^{i\alpha}\right\rangle_{pq}$ consistent with zero
- With anti-periodic BCs and all else the same $\left\langle e^{i\alpha}\right\rangle_{pq} \approx 1$
  $\longrightarrow$ phase reweighting not even necessary

Even stranger

Other $\langle\mathcal{O}\rangle_{pq}$ nearly identical
despite sign problem...

Can this be understood?



Numerical results for volume & N dependence

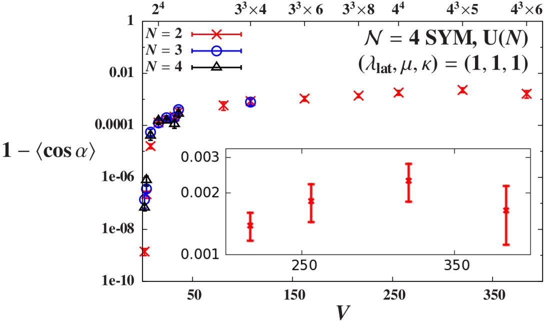
No indication of a sign problem with anti-periodic BCs

- Pfaffian $P = |P|e^{i\alpha}$ is nearly real and positive, $1 - \langle\cos(\alpha)\rangle \ll 1$
- Fluctuations in pfaffian phase don't grow with the lattice volume
- Insensitive to number of colors $N = 2, 3, 4$

Hard calculations

Each $4^3 \times 6$ measurement requires $\sim$8 days, $\sim$10GB memory

Parallel $\mathcal{O}(n^3)$ algorithm



Thank you!

Thank you!

Collaborators

Simon Catterall, Poul Damgaard, Tom DeGrand and Joel Giedt

Funding and computing resources

Backup: Failure of Leibnitz rule on lattice

Given that $\left\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\right\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = 2i\sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu$ is problematic,

why not try $\left\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\right\} = 2i\sigma^\mu_{\alpha\dot{\alpha}} \nabla_\mu$ for a discrete translation?

Here $\nabla_\mu \phi(x) = \frac{1}{a}\left[\phi(x + a\widehat{\mu}) - \phi(x)\right] = \partial_\mu \phi(x) + \frac{a}{2}\partial_\mu^2 \phi(x) + \mathcal{O}(a^2)$

Essential difference between ∂_μ and ∇_μ on the lattice ($a > 0$)

$$\begin{aligned}\nabla_\mu \left[\phi(x)\chi(x)\right] &= a^{-1}\left[\phi(x + a\widehat{\mu})\chi(x + a\widehat{\mu}) - \phi(x)\chi(x)\right] \\ &= \nabla_\mu \left[\phi(x)\right]\chi(x) + \phi(x)\nabla_\mu \chi(x) + a\nabla_\mu \left[\phi(x)\right]\nabla_\mu \chi(x)\end{aligned}$$

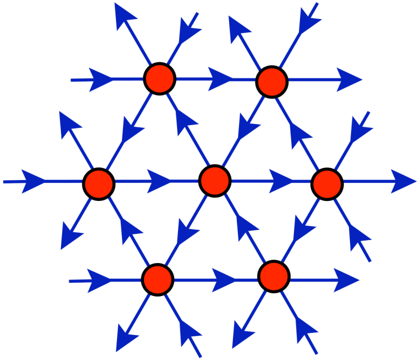
We only recover the Leibnitz rule $\partial_\mu(fg) = \partial_\mu(f)g + f\partial_\mu g$ when $a \to 0$

$\Longrightarrow$ “Discrete supersymmetry” breaks down on the lattice

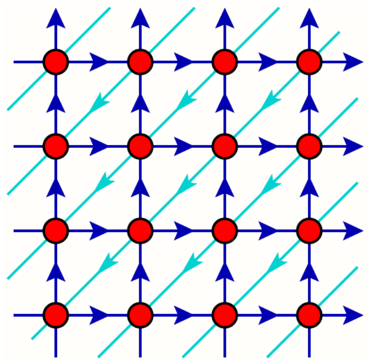
(Dondi & Nicolai, “Lattice Supersymmetry”, 1977)

Backup: Hypercubic basis for A_4^* lattice

It is very convenient to represent the A_4^* lattice
as a hypercube with a backwards diagonal



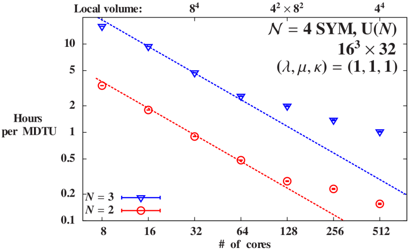
$\rightsquigarrow$

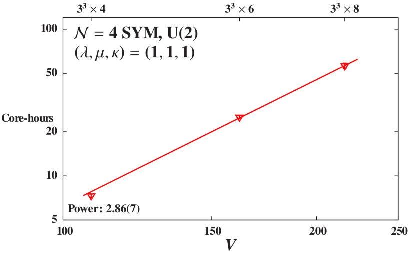


Backup: Code performance—weak and strong scaling

Left: Strong scaling for U(2) and U(3) $16^3 \times 32$ RHMC

Right: Weak scaling for $\mathcal{O}(N_\Psi^3)$ pfaffian calculation (fixed local volume)



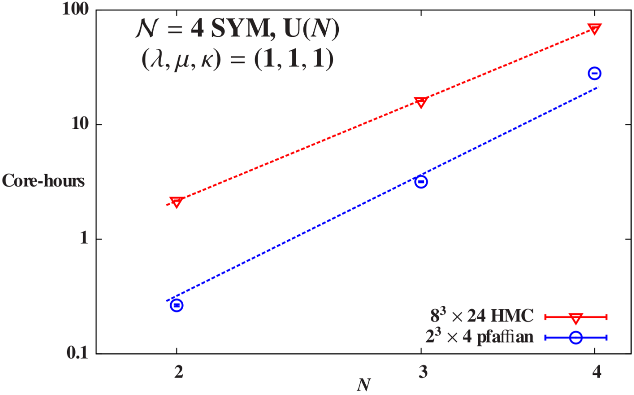


Both plots on log–log axes with power-law fits

Backup: Code performance for 2, 3 and 4 colors

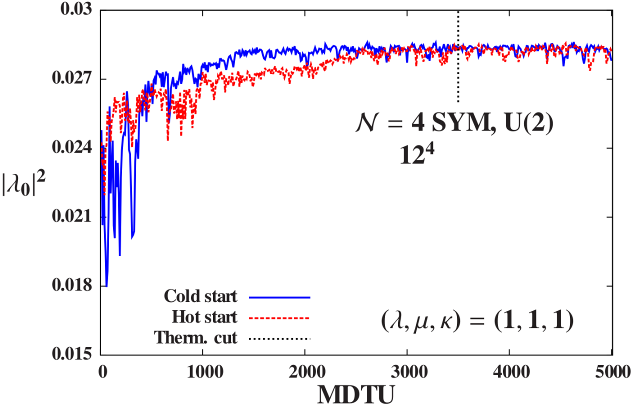
Red: Find RHMC costs scaling $\sim N^5$ (recall $16N^2$ fermion components)

Blue: Pfaffian costs consistent with expected N^6 scaling



Backup: Thermalization

Thermalization becomes increasingly painful as N and L increase
Example: Evolution of smallest $\mathcal{D}^\dagger\mathcal{D}$ eigenvalue $|\lambda_0|^2$



Shouldn't be too hard to address this with better initial configuration

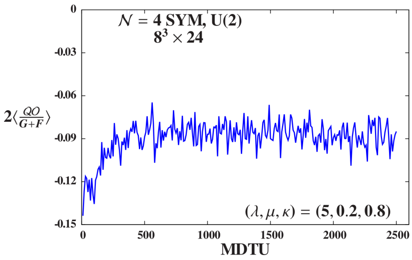
Backup: The problem with flat directions

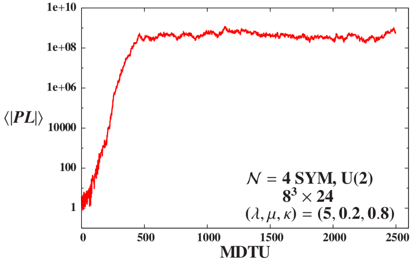
Gauge fields can move far away from $\mathcal{U}_a \longrightarrow \mathbb{I} + \mathcal{A}_a$
if $N\mu/(2\lambda_{\text{lat}})$ becomes too small

Example for two-color $(\lambda_{\text{lat}}, \mu, \kappa) = (5, 0.2, 0.8)$ on $8^3 \times 24$ volume

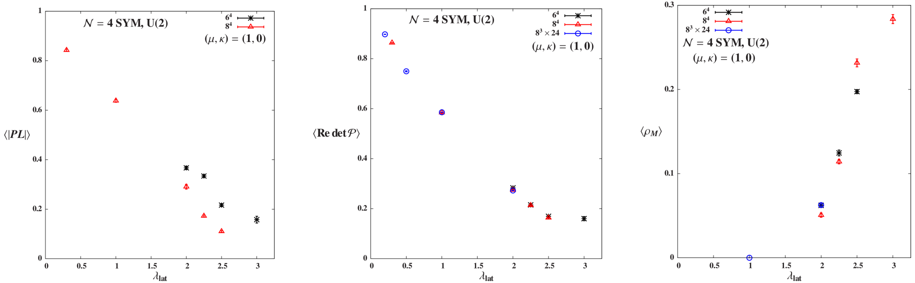
Left: Ward identity violations are stable at $\sim$10% level

Right: Polyakov loop wanders off to $\sim 10^9$





Backup: Lattice phase due to U(1) sector



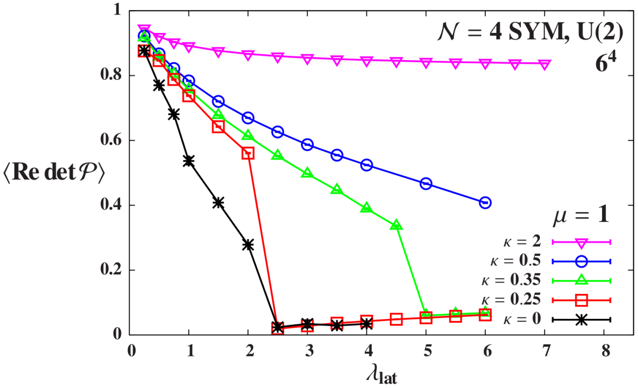
1. Polyakov loop collapses $\Longrightarrow$ confining phase
   (**not** present in continuum $\mathcal{N} = 4$ SYM)
2. Plaquette determinant is variable in U(1) sector
   Drops at same coupling λ as Polyakov loop
3. ρ_M is density of U(1) monopole world lines (DeGrand & Toussaint)
   Non-zero when Polyakov loop and plaq. determinant collapse

Backup: Suppressing the U(1) sector

$\Delta S = \kappa |\det \mathcal{P} - 1|^2$ suppresses the strongly-coupled lattice phase

Produces $2\kappa F_{\mu\nu} F^{\mu\nu}$ term in U(1) sector

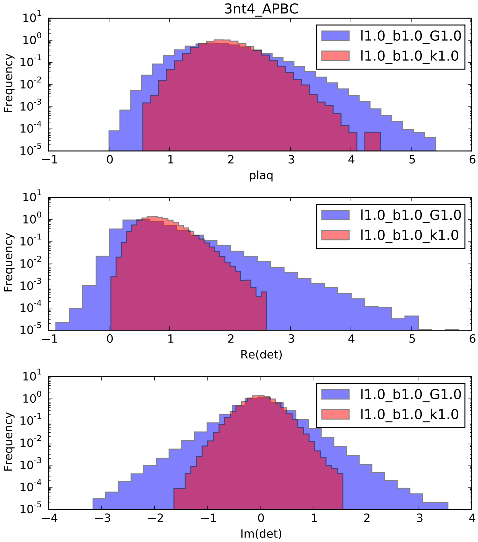
$\Longrightarrow$ QED critical $\beta_c = 0.99 \longrightarrow$ critical $\kappa_c \approx 0.5$



Backup: Plaquette and determinant distributions

Price of $\mathcal{Q}$-invariant determinant deformation:

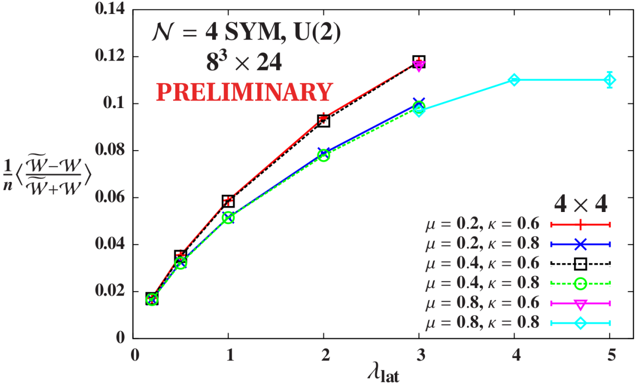
Distributions tend to broaden out, at least for parameters currently in use



Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

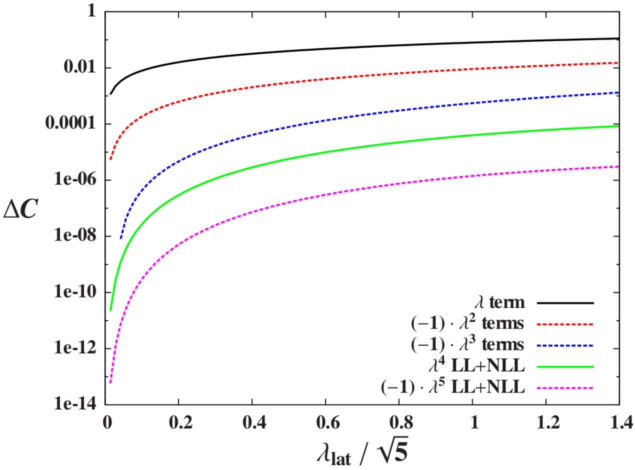
Restoration of the other 15 $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ in the continuum limit
follows from restoration of R symmetry (motivation for A_4^* lattice)

Modified Wilson loops test R symmetries at non-zero lattice spacing



Backup: Perturbation theory for Coulomb coefficient

For range of λ_{lat} currently being studied, the perturbative series for the U(3) Coulomb coefficient appears well convergent

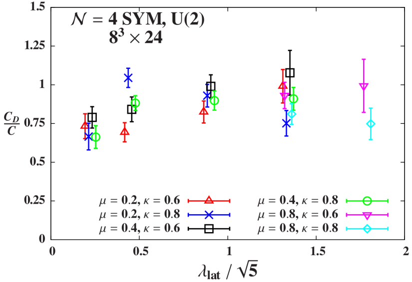


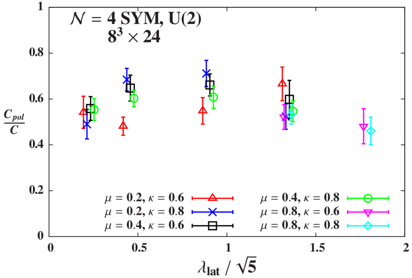
Backup: More tests of the U(2) static potential

Left: Projecting Wilson loops from U(2) $\longrightarrow$ SU(2)

$$\Longrightarrow \text{factor of } \frac{N^2-1}{N^2} = 3/4$$

Right: Unitarizing links removes scalars $\Longrightarrow$ factor of $1/2$





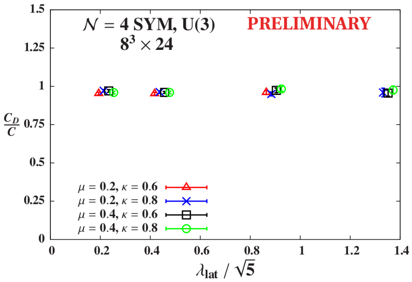
Both expected factors present, although (again) noisily

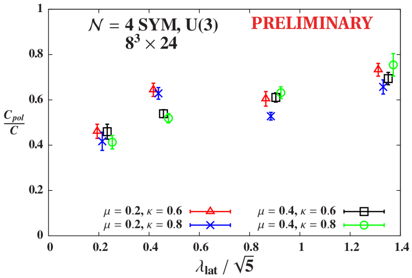
Backup: More tests of the U(3) static potential

Left: Projecting Wilson loops from U(3) $\longrightarrow$ SU(3)

$\Longrightarrow$ factor of $\frac{N^2-1}{N^2} = 8/9$

Right: Unitarizing links removes scalars $\Longrightarrow$ factor of $1/2$



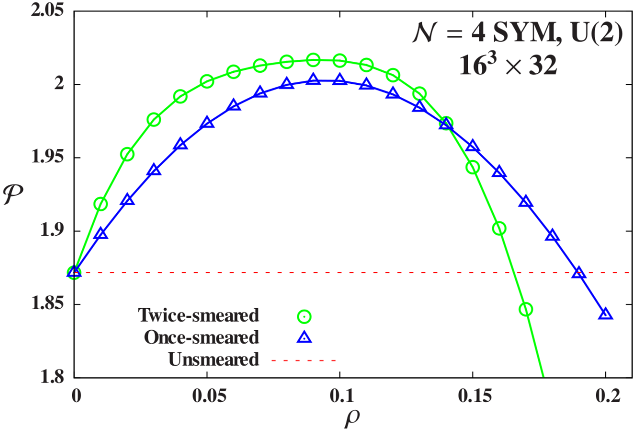


Ratios look slightly higher than expected,

less noise in SU(3)-projected results

Backup: Smearing for noise reduction

Smearing may reduce noise in static potential (etc.) measurements
—Stout smearing implemented and tested
—APE or HYP (without unitary projection) may work better for Konishi



Backup: Pfaffian phase dependence on λ_{lat}, μ, κ

We observe little dependence on κ

Fluctuations in phase grow as λ_{lat} increases