Maximally supersymmetric Yang–Mills on the lattice

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with Simon Catterall, Poul Damgaard, Tom DeGrand and Joel Giedt
Context: Why lattice supersymmetry

At strong coupling...

—Supersymmetric gauge theories are particularly interesting:
  Dualities, holography, confinement, conformality, . . .

—Nonperturbative lattice discretization is particularly useful
  Numerical analysis provides complementary approach to SCGT

Proven success for QCD; many potential susy applications:

- Compute Wilson loops, spectrum, scaling dimensions, etc.,
  complementing perturbation theory, holography, bootstrap, . . .

- Further direct checks of conjectured dualities

- Predict low-energy constants from dynamical susy breaking

- Validate or refine AdS/CFT-based modelling
  (e.g., QCD phase diagram, condensed matter systems)
There is a problem with supersymmetry in discrete space-time.

Recall: supersymmetry extends Poincaré symmetry by spinorial generators $Q^I_\alpha$ and $\overline{Q}^I_{\dot{\alpha}}$ with $I = 1, \cdots, \mathcal{N}$.

The resulting algebra includes $\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2\sigma^{\mu}_{\alpha\dot{\alpha}} P_\mu$.

$P_\mu$ generates infinitesimal translations, which don’t exist on the lattice $\implies$ supersymmetry explicitly broken at classical level.

Consequence for lattice calculations:

Quantum effects generate (typically many) susy-violating operators.

Fine-tuning their couplings to restore susy is generally not practical.
Exact susy on the lattice: $\mathcal{N} = 4$ SYM

In order to forbid generation of susy-violating operators
(some subset of) the susy algebra must be preserved

In four dimensions $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM)
is the only known system with a supersymmetric lattice formulation

$\mathcal{N} = 4$ SYM is a particularly interesting theory

- SU($N$) gauge theory with four fermions $\psi^I$ and six scalars $\phi^{IJ}$, all massless and in adjoint rep.

- Action consists of kinetic, Yukawa and four-scalar terms

- Supersymmetric: 16 supercharges $Q_I^\alpha$ and $\overline{Q}_{\dot{\alpha}}^I$ with $I = 1, \ldots, 4$
  Fields and $Q$'s transform under global $SU(4) \simeq SO(6)$ $R$ symmetry

- Conformal: $\beta$ function is zero for all 't Hooft couplings $\lambda$
Exact susy on the lattice: topological twisting

What is special about $\mathcal{N} = 4$ SYM

The 16 fermionic supercharges $Q^I_\alpha$ and $\bar{Q}^I_{\dot{\alpha}}$ fill a Kähler–Dirac multiplet:

$$
\begin{pmatrix}
Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\
\bar{Q}^1_{\dot{\alpha}} & \bar{Q}^2_{\dot{\alpha}} & \bar{Q}^3_{\dot{\alpha}} & \bar{Q}^4_{\dot{\alpha}}
\end{pmatrix} = Q + \gamma_\mu Q_\mu + \gamma_\mu \gamma_\nu Q_{\mu\nu} + \gamma_\mu \gamma_5 Q_{\mu\nu\rho} + \gamma_5 Q_{\mu\nu\rho\sigma}
\longrightarrow Q + \gamma_a Q_a + \gamma_a \gamma_b Q_{ab}
$$

with $a, b = 1, \cdots, 5$

This is a decomposition in representations of a “twisted rotation group”

$$SO(4)_{tw} \equiv \text{diag} \left[ SO(4)_{\text{euc}} \otimes SO(4)_R \right]$$

$$SO(4)_R \subset SO(6)_R$$

In this notation there is a susy subalgebra $\{Q, Q\} = 2Q^2 = 0$

This can be exactly preserved on the lattice
Twisted $\mathcal{N} = 4$ SYM

$$\text{SO}(4)_{tw} \equiv \text{diag} \left[ \text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R \right]$$

- $Q$, $Q_\mu$, $Q_{\mu\nu}$, ... transform with **integer spin** – no longer spinors!
- Fermion fields decompose in the same way, $\psi^I \rightarrow \{\eta, \psi_a, \chi_{ab}\}$
- Scalar fields transform as $\text{SO}(4)_{tw}$ vector $B_\mu$ plus two scalars $\phi$, $\bar{\phi}$
  Combine with $A_\mu$ in complexified five-component gauge field

$$A_a = A_a + iB_a = (A_\mu, \phi) + i(B_\mu, \bar{\phi})$$

and similarly for $\overline{A}_a$

Complexified gauge field $\rightarrow U(N) = SU(N) \otimes U(1)$ gauge invariance

Irrelevant in the continuum, but will affect lattice calculations
Twisted $\mathcal{N} = 4$ SYM

$$\text{SO}(4)_{\text{tw}} \equiv \text{diag}\left[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_{\text{R}}\right]$$

- $Q, Q_\mu, Q_{\mu\nu}, \ldots$ transform with **integer spin** – no longer spinors!
- Fermion fields decompose in the same way, $\psi^I \rightarrow \{\eta, \psi_a, \chi_{ab}\}$
- Scalar fields transform as $\text{SO}(4)_{\text{tw}}$ vector $B_\mu$ plus two scalars $\phi, \phi$ Combine with $A_\mu$ in complexified five-component gauge field

$$A_a = A_a + iB_a = (A_\mu, \phi) + i(B_\mu, \phi) \quad \text{and similarly for } \bar{A}_a$$

In flat space twisting is just a change of variables, no effect on physics

Same lattice system also results from orbifolding / dimensional deconstruction approach
Now we can move directly to the lattice

Twisting gives manifestly supersymmetric lattice action for $\mathcal{N} = 4$ SYM

$$S = \frac{N}{2\lambda_{\text{lat}}} Q \left( \chi_{ab} F_{ab} + \eta \overline{D} a U_a - \frac{1}{2} \eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde} \chi_{ab} \overline{D} c \chi_{de}$$

$QS = 0$ follows from $Q^2 \cdot = 0$ and Bianchi identity.

- We have exact $U(N)$ gauge invariance
- We exactly preserve $Q$, one of 16 supersymmetries
- Restoration of twisted $SO(4)_{tw}$ in continuum limit guarantees recovery of other 15 $Q_a$ and $Q_{ab}$

The theory is almost suitable for practical numerical calculations...
Stabilizing numerical calculations

We need to add two deformations to the $Q$-invariant action

Both deal with features required by the supersymmetric construction

**Scalar potential to regulate flat directions**

Gauge links $U_a$ must be elements of algebra, like fermions

$\rightarrow$ Add scalar potential \( \left( \frac{1}{N} \text{Tr} \left[ U_a \overline{U}_a \right] - 1 \right)^2 \) to lift flat directions

Otherwise $U_a$ can wander far from continuum form $U_a = \mathbb{I}_N + A_a$

**Plaquette determinant to suppress $U(1)$ sector of $U(N)$**

$U_a$ complexified $\rightarrow$ Add approximate SU($N$) projection $|\text{det} \mathcal{P}_{ab} - 1|^2$

where $\mathcal{P}_{ab}$ is the product of four $U_a$ around the elementary plaquette

Otherwise encounter strong-coupling $U(1)$ confinement transition
Soft susy breaking from naive stabilization

Directly adding scalar potential and plaquette determinant to action explicitly breaks supersymmetry

\[ S = \frac{N}{2\lambda_{\text{lat}}} Q \left( \chi_{ab} F_{ab} + \eta \overline{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde} \chi_{ab} \overline{D}_c \chi_{de} \\
+ \frac{N}{2\lambda_{\text{lat}}} \mu^2 \left( \frac{1}{N} \text{Tr} \left[ U_a \overline{U}_a \right] - 1 \right)^2 + \kappa |\text{det} P_{ab} - 1|^2 \]

Breaking is **soft**

Guaranteed to vanish as \( \mu, \kappa \rightarrow 0 \)

Also suppressed \( \propto 1/N^2 \)

1–10% effects in practice
New development: Supersymmetric stabilization

Possible to construct $Q$-invariant scalar potential and plaquette det.

However, these result in positive vacuum energy (non-susy)

$$S = \frac{N}{2\lambda_{\text{lat}}} Q \left( \chi_{ab} \mathcal{F}_{ab} + \eta \{ \bar{D}_a U_a + X \} - \frac{1}{2} \eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde} \chi_{ab} \bar{D}_c \chi_{de}$$

$$X = B^2 \left( \frac{1}{N} \text{Tr} [U_a \bar{U}_a] - 1 \right)^2 + G |\text{det} \mathcal{P}_{ab} - 1|^2$$

Again effects vanish as $B, G \rightarrow 0$

Allows access to much stronger $\lambda$ with much smaller artifacts
Final thought on the lattice $\mathcal{N} = 4$ SYM formulation

The construction is obviously very complicated

(For experts: $\gtrsim 100$ inter-node data transfers in the fermion operator)

To reduce this barrier to others entering the field, we make our efficient parallel code publicly available

github.com/daschaich/susy

Evolved from MILC lattice QCD code, presented in arXiv:1410.6971 — CPC appeared yesterday
Physics result: Static potential is Coulombic at all $\lambda$

Static potential $V(r)$ from $r \times T$ Wilson loops: $W(r, T) \propto e^{-V(r) T}$

Fit $V(r)$ to Coulombic or confining form

$$V(r) = A - C/r$$

$$V(r) = A - C/r + \sigma r$$

Fits to confining form always produce vanishing string tension $\sigma = 0$

Working on standard methods to reduce noise
Coupling dependence of $V(r) = A - C/r$

—Perturbation theory predicts $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

—AdS/CFT predicts $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$, $\lambda \to \infty$, $\lambda \ll N$

We see agreement with perturbation theory for $N = 2$, $\lambda \lesssim 2$, and a tantalizing discrepancy for $N = 3$, $\lambda \gtrsim 1$

No dependence on $\mu$ or $\kappa$ $\longrightarrow$ apparently insensitive to soft $Q$ breaking
Recapitulation

- Strongly coupled supersymmetric field theories very interesting to study through lattice calculations

- Practical numerical calculations possible for lattice $\mathcal{N} = 4$ SYM based on exact preservation of twisted susy subalgebra $Q^2 = 0$

- The construction is complicated
  $\longrightarrow$ publicly-available code to reduce barriers to entry

- The static potential is always Coulombic
  For $N = 2$ $C(\lambda)$ is consistent with perturbation theory
  For $N = 3$ an intriguing discrepancy at stronger couplings

- There are many more directions to pursue in the future
  ▶ Measuring anomalous dimension of Konishi operator
  ▶ Understanding the (absence of a) sign problem
Thank you!
Thank you!

Collaborators
Simon Catterall, Poul Damgaard, Tom DeGrand and Joel Giedt

Funding and computing resources
Recall $\mathcal{N} = 4$ SYM is conformal

\[ \Rightarrow \text{All correlation functions decay algebraically } \propto r^{-\Delta} \]

The Konishi operator is the simplest conformal primary operator

\[ \mathcal{O}_K = \sum_I \text{Tr} [\Phi^I \Phi^I] \quad C_K(r) \equiv \mathcal{O}_K(x + r)\mathcal{O}_K(x) = Ar^{-2\Delta_K} \]

There are many predictions for the scaling dim. $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$

- From perturbation theory for small $\lambda$,
  related to $\lambda \to \infty$ by S duality under $\frac{4\pi N}{\lambda} \leftrightarrow \frac{\lambda}{4\pi N}$

- From holography for $N \to \infty$ and $\lambda \to \infty$ but $\lambda \ll N$

- Bounds on max $\{\Delta_K\}$ from the conformal bootstrap program

We will add lattice gauge theory to this list
Konishi operator on the lattice

\[ \mathcal{O}_K = \sum \text{Tr} [\Phi^I \Phi^I] \rightarrow \hat{\mathcal{O}}_K = \sum \text{Tr} [\hat{\varphi}^a \hat{\varphi}^b] \]

with \( \hat{\varphi}^a = U_a \bar{U}_a - \frac{1}{N} \text{Tr} [U_a \bar{U}_a] \)

\[ C(r) = \hat{\mathcal{O}}_K(x + r)\hat{\mathcal{O}}_K(x) \propto r^{-2\Delta_K} \]

Consistent with power laws using perturbative \( \Delta \)

Need \( Q \)-invariant plaquette det. for reasonable \( C(r) \) on \( 8^4 \) lattice

Obviously not a stable way to determine \( \Delta_K \) — we have other tools
Preliminary Konishi $\Delta_K$ from Monte Carlo RG

Scaling dimension is eigenvalue of MCRG “stability matrix”

Simple trial (only statistical errors) correctly finds $\Delta_K \to 2$ as $\lambda \to 0$

Significant volume dependence $\rightarrow$ approach perturbation theory as $L$ increases

Need to check systematics:
- different numbers of blocking steps, different operators, different $G$

Need to produce consistent results from independent approach(es) such as finite-size scaling
Supplement: The (absence of a) sign problem

In lattice gauge theory we compute operator expectation values

\[
\langle O \rangle = \frac{1}{Z} \int [dU][d\bar{U}] O e^{-S_B[U,\bar{U}]} \text{pf} D[U,\bar{U}]
\]

\(\text{pf} D = |\text{pf} D| e^{i\alpha}\) is generically complex for lattice \(\mathcal{N} = 4\) SYM

\(\rightarrow\) Complicates interpretation of \([e^{-S_B} \text{pf} D]\) as Boltzmann weight

Have to reweight "phase-quenched" (pq) calculations

\[
\langle O \rangle_{pq} = \frac{1}{Z_{pq}} \int [dU][d\bar{U}] O e^{-S_B[U,\bar{U}]} |\text{pf} D| \quad \langle O \rangle = \frac{\langle O e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}}
\]

Sign problem: This breaks down if \(\langle e^{i\alpha} \rangle_{pq}\) is consistent with zero
Illustration of sign problem and its absence

- With **periodic temporal fermion boundary conditions** we have an obvious sign problem, $\langle e^{i\alpha} \rangle_{pq}$ consistent with zero.

- With **anti-periodic BCs** and all else the same $\langle e^{i\alpha} \rangle_{pq} \approx 1$ → phase reweighting not even necessary.

Even stranger

Other $\langle O \rangle_{pq}$ nearly identical despite sign problem...

Can this be understood?
No indication of a sign problem with anti-periodic BCs

- Pfaffian $P = |P|e^{i\alpha}$ is nearly real and positive, $1 - \langle \cos(\alpha) \rangle \ll 1$
- Fluctuations in pfaffian phase don’t grow with the lattice volume
- Insensitive to number of colors $N = 2, 3, 4$

Hard calculations

Each $4^3 \times 6$ measurement requires $\sim 8$ days, $\sim 10$GB memory

Parallel $O(n^3)$ algorithm
Backup: Failure of Leibnitz rule in discrete space-time

Given that \( \{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \} = 2 \sigma_{\alpha \dot{\alpha}}^\mu P_\mu = 2i \sigma_{\alpha \dot{\alpha}}^\mu \partial_\mu \) is problematic, why not try \( \{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \} = 2i \sigma_{\alpha \dot{\alpha}}^\mu \nabla_\mu \) for a discrete translation?

Here \( \nabla_\mu \phi(x) = \frac{1}{a} [\phi(x + a\hat{\mu}) - \phi(x)] = \partial_\mu \phi(x) + \frac{a}{2} \partial_\mu^2 \phi(x) + O(a^2) \)

Essential difference between \( \partial_\mu \) and \( \nabla_\mu \) on the lattice (\( a > 0 \))

\[
\nabla_\mu [\phi(x)\chi(x)] = a^{-1} [\phi(x + a\hat{\mu})\chi(x + a\hat{\mu}) - \phi(x)\chi(x)] \\
= [\nabla_\mu \phi(x)] \chi(x) + \phi(x) \nabla_\mu \chi(x) + a [\nabla_\mu \phi(x)] \nabla_\mu \chi(x)
\]

We only recover the Leibnitz rule \( \partial_\mu (fg) = (\partial_\mu f)g + f\partial_\mu g \) when \( a \to 0 \)

\( \implies \) “Discrete supersymmetry” breaks down on the lattice

(Dondi & Nicolai, “Lattice Supersymmetry”, 1977)
The Kähler–Dirac representation is related to the usual $Q^I_\alpha$, $\bar{Q}^I_{\dot{\alpha}}$ by

$$
\begin{pmatrix}
Q_1^\alpha & Q_2^\alpha & Q_3^\alpha & Q_4^\alpha \\
\bar{Q}_1^{\dot{\alpha}} & \bar{Q}_2^{\dot{\alpha}} & \bar{Q}_3^{\dot{\alpha}} & \bar{Q}_4^{\dot{\alpha}}
\end{pmatrix}
= Q + \gamma_\mu Q_\mu + \gamma_\mu \gamma_\nu Q_{\mu\nu} + \gamma_\mu \gamma_5 Q_{\mu\nu\rho} + \gamma_5 Q_{\mu\nu\rho\sigma}

\rightarrow Q + \gamma_a Q_a + \gamma_a \gamma_b Q_{ab}
$$

with $a, b = 1, \cdots, 5$

The 4 $\times$ 4 matrix involves R symmetry transformations along each row and (euclidean) Lorentz transformations along each column

$\rightarrow$ Kähler–Dirac components transform under “twisted rotation group”

$$
SO(4)_{tw} \equiv \text{diag} \left[ SO(4)_{\text{euc}} \otimes SO(4)_R \right]
$$

only $SO(4)_R \subset SO(6)_R$
Backup: Details of $Q^2 = 0$ on the lattice

**Goal:** Preserve $Q$ supersymmetry on the lattice

1. $Q^2 \cdot = 0$
2. $Q$ directly interchanges bosonic $\leftrightarrow$ fermionic d.o.f.

Both conditions are easy to verify in five-component notation:

- $Q \ U_a = \psi_a$
- $Q \ \chi_{ab} = -\bar{F}_{ab}$
- $Q \ \eta = d$

- $Q \ \psi_a = 0$
- $Q \ \bar{U}_a = 0$
- $Q \ d = 0$

- Gauge field $U_a$ and $\psi_a$ live on links between lattice sites
  - $U_a$ must be elements of algebra $\mathfrak{gl}(N, \mathbb{C}) \ni \psi_a$
  - $\implies$ Non-trivial to ensure $U_a \to I + A_a$ in the continuum limit

- Field strength $\bar{F}_{ab}$ and $\chi_{ab}$ live on diagonals of oriented faces

- Bosonic auxiliary field $d$ and $\eta$ live on sites
  - Usual equation of motion: $d = \overline{D}_a U_a$
Backup: $A_4^*$ lattice with five links in four dimensions

$A_a = (A_\mu, \phi)$ may remind you of dimensional reduction

On the lattice we need to treat all five $U_a$ symmetrically

—Start with hypercubic lattice
  in 5d momentum space

—**Symmetric** constraint $\sum_a \partial_a = 0$
  projects to 4d momentum space

—Result is $A_4$ lattice
  $\rightarrow$ dual $A_4^*$ lattice in real space
Backup: Twisted SO(4) symmetry on the $A_4^*$ lattice

—Can picture $A_4^*$ lattice as 4d analog of 2d triangular lattice

—Five basis vectors are non-orthogonal and linearly dependent

—Preserves $S_5$ point group symmetry

$S_5$ irreps precisely match onto irreps of twisted SO(4)$_{tw}$

\[
\begin{align*}
5 &= 4 \oplus 1 : & U_a &\rightarrow A_\mu, \quad \phi \\
& & \psi_a &\rightarrow \psi_\mu, \quad \eta_{\mu\nu\rho\sigma} \\
10 &= 6 \oplus 4 : & \chi_{ab} &\rightarrow \chi_{\mu\nu}, \quad \psi_{\mu\nu\rho}
\end{align*}
\]
Backup: Analytic results for exact lattice susy

\[ S = \frac{N}{2\lambda_{\text{lat}}} Q \left( \chi_{ab} F_{ab} + \eta \bar{D} a U a - \frac{1}{2} \eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde} \chi_{ab} \bar{D} c \chi_{de} \]

Gauge invariant, \( Q \) supersymmetric, \( S_5 \) symmetric

The high degree of symmetry has important consequences

- Moduli space preserved to all orders of lattice perturbation theory
  \( \rightarrow \) no scalar potential induced by radiative corrections

- \( \beta \) function vanishes at one loop (at least)

- Real-space RG blocking transformations preserve \( Q \) & \( S_5 \)

- Only one marginal tuning to recover \( Q_a \) and \( Q_{ab} \) in the continuum
Backup: Hypercubic basis for $A_4^*$ lattice

It is very convenient to represent the $A_4^*$ lattice as a hypercube with a backwards diagonal.
Backup: Code performance—weak and strong scaling

**Left:** Strong scaling for U(2) and U(3) $16^3 \times 32$ RHMC

**Right:** Weak scaling for $O(N^3_\psi)$ pfaffian calculation (fixed local volume)

$N_\psi \equiv 16N^2L^3N_T$ is number of fermion degrees of freedom

Both plots on log–log axes with power-law fits
Backup: Code performance for 2, 3 and 4 colors

**Red:** Find RHMC costs scaling $\sim N^5$ (recall adjoint fermion d.o.f. $\propto N^2$)

**Blue:** Pfaffian costs consistent with expected $N^6$ scaling
Backup: Thermalization

Thermalization becomes increasingly painful as $N$ or $L^3 \times N_T$ increase.

Example: Evolution of smallest $\mathcal{D}^\dagger \mathcal{D}$ eigenvalue $|\lambda_0|^2$

Should be possible to address this with better initial configuration.
Backup: The problem with flat directions

Gauge fields $\mathcal{U}_a$ can move far away from continuum form $\mathbb{I} + A_a$ if $N\mu^2/(2\lambda_{\text{lat}})$ becomes too small.

Example for two-color $(\lambda_{\text{lat}}, \mu, \kappa) = (5, 0.2, 0.8)$ on $8^3 \times 24$ volume

**Left:** Ward identity violations are stable at $\sim 9\%$

**Right:** Polyakov loop wanders off to $\sim 10^9$
Backup: Lattice phase due to U(1) sector

1. Polyakov loop collapses $\Rightarrow$ confining phase (not present in continuum $\mathcal{N} = 4$ SYM)

2. Plaquette determinant is variable in U(1) sector
   Drops at same coupling $\lambda$ as Polyakov loop

3. $\rho_M$ is density of U(1) monopole world lines (DeGrand & Toussaint)
   Non-zero when Polyakov loop and plaquette det. collapse
\[ \Delta S = \kappa |\det P - 1|^2 \] suppresses the lattice strong-coupling phase

Produces \( 2\kappa F_{\mu\nu} F^{\mu\nu} \) term in U(1) sector

\[ \Rightarrow \text{QED critical } \beta_c = 0.99 \rightarrow \text{critical } \kappa_c \approx 0.5 \]
Backup: Plaquette and determinant distributions

Larger couplings $B$ and $G$ produce the desired sharper peaks

Price: Larger Ward identity violations and larger computational costs
Backup: Restoration of $Q_a$ and $Q_{ab}$ supersymmetries

Restoration of the other 15 $Q_a$ and $Q_{ab}$ in the continuum limit follows from restoration of $R$ symmetry (motivation for $A_4^*$ lattice)

Modified Wilson loops test $R$ symmetries at non-zero lattice spacing

$\mathcal{N} = 4$ SYM, U(2)

$8^3 \times 24$

PRELIMINARY

$\frac{1}{n} \langle \tilde{W} - W \rangle \overline{W + W'}$

$\lambda_{lat}$

$\mu = 0.2$, $\kappa = 0.6$
$\mu = 0.2$, $\kappa = 0.8$
$\mu = 0.4$, $\kappa = 0.6$
$\mu = 0.4$, $\kappa = 0.8$
$\mu = 0.8$, $\kappa = 0.6$
$\mu = 0.8$, $\kappa = 0.8$
Backup: $\mathcal{N} = 4$ static potential from Wilson loops

Extract static potential $V(r)$ from $r \times T$ Wilson loops: $W(r, T) \propto e^{-V(r)T}$

Coulomb gauge trick from lattice QCD reduces $A^*_4$ lattice complications
For range of $\lambda_{\text{lat}}$ currently being studied, the perturbative series for the U(3) Coulomb coefficient appears well convergent.
Backup: More tests of the $U(2)$ static potential

**Left:** Projecting Wilson loops from $U(2) \rightarrow SU(2)$

$$\Rightarrow \text{factor of } \frac{N^2 - 1}{N^2} = \frac{3}{4}$$

**Right:** Unitarizing links removes scalars $\Rightarrow$ factor of $\frac{1}{2}$

Both expected factors present, although (again) noisily
Backup: More tests of the U(3) static potential

**Left:** Projecting Wilson loops from U(3) $\rightarrow$ SU(3)

$\Rightarrow$ factor of $\frac{N^2 - 1}{N^2} = \frac{8}{9}$

**Right:** Unitarizing links removes scalars $\Rightarrow$ factor of $\frac{1}{2}$

Ratios look slightly higher than expected, less noise in SU(3)-projected results
Backup: Smearing for noise reduction

Smearing may reduce noise in static potential (etc.) measurements
—Stout smearing implemented and tested
—APE or HYP (without unitary projection) may work better for Konishi
Backup: Konishi operator on the lattice

$$\mathcal{O}_K = \sum_i \text{Tr} \left[ \Phi^I \Phi^I \right]$$

On the lattice the scalars $\Phi^I$ are twisted and wrapped up in the complexified gauge field $\mathcal{U}_a$

Given $\mathcal{U}_a \approx \mathbb{I} + A_a$ the most obvious way to extract the scalars is

$$\hat{\varphi}^a = \mathcal{U}_a \overline{\mathcal{U}}_a - \frac{1}{N} \text{Tr} \left[ \mathcal{U}_a \overline{\mathcal{U}}_a \right] \mathbb{I}$$

This is still twisted, so all $\{a, b\}$ contribute to R-singlet Konishi

$$\hat{\mathcal{O}}_K = \sum_{a, b} \text{Tr} \left[ \hat{\varphi}^a \hat{\varphi}^b \right]$$
Couplings flow under RG blocking transformation $R_b$

$n$-times-blocked system is $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} O_i^{(n)}$

Consider linear expansion around fixed point $H^*$ with couplings $c_i^*$

$$c_i^{(n)} - c_i^* = \sum_j \frac{\partial c_i^{(n)}}{\partial c_j^{(n-1)}} \left|_{H^*} \right. \left( c_j^{(n-1)} - c_j^* \right) \equiv \sum_j T_{ij}^* \left( c_j^{(n-1)} - c_j^* \right)$$

$T_{ij}^*$ is the “stability matrix”

Eigenvalues of $T_{ij}^*$ are scaling dimensions of corresponding operators
We observe little dependence on $\kappa$

Fluctuations in phase grow as $\lambda_{\text{lat}}$ increases.

\[ N = 4 \text{ SYM, } U(2) \]

\[ (\lambda, \mu) = (1, 1) \]

\[ \kappa = 0.6, 1.0, 2.0 \]