

# Fun with the $S$ parameter on the lattice

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from work with the [LSD Collaboration](#) and [USQCD BSM community](#)  
[arXiv:1009.5967](#) & [arXiv:1111.4993](#)

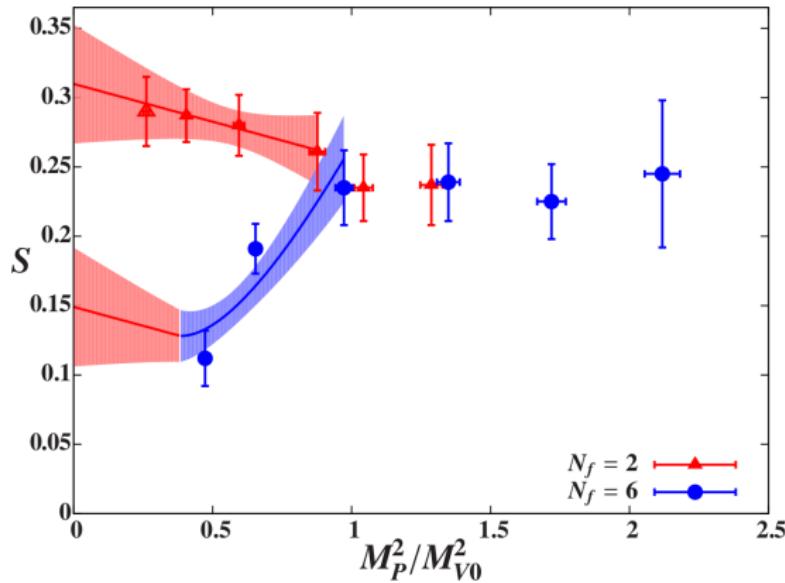
Origin of Mass 2013 — Lattice BSM Workshop, Odense, 7 August



# Overview

You saw this plot yesterday

Today I informally summarize how I made it



(From the 2013 USQCD white paper

*Lattice Gauge Theories at the Energy Frontier*)

## $S$ on the lattice: definitions

$$S = 4\pi \textcolor{red}{N}_D \lim_{Q^2 \rightarrow 0} \frac{d}{dQ^2} \Pi_{V-A}(Q^2) - \Delta S_{SM}(M_H)$$



- ➊  $\Pi_{V-A}^{\mu\nu}(Q) = Z \sum_x e^{iQ \cdot (x + \hat{\mu}/2)} \text{Tr} \left[ \langle V^{\mu a}(x) V^{\nu b}(0) \rangle - \langle A^{\mu a}(x) A^{\nu b}(0) \rangle \right]$   
(correlators mix conserved and local domain wall currents for efficiency)
- ➋  $\textcolor{red}{N}_D \geq 1$  is the number of doublets with chiral electroweak couplings
- ➌  $\Delta S_{SM}(M_H)$  subtracted so that  $S = 0$  in the standard model  
Removes three eaten Goldstones, depends on Higgs mass

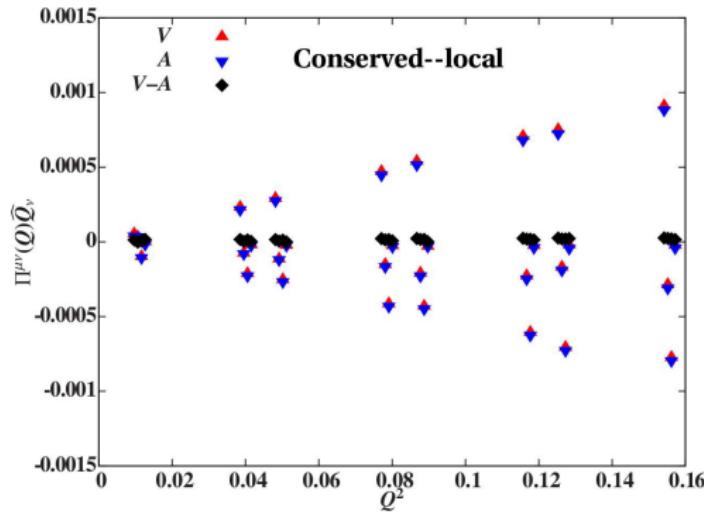
—Results below (from arXiv:1009.5967) take  $M_H \sim 1000$  GeV  
—This morning: May not need to shift all the way to 125 GeV

# Cancellation of lattice artifacts in mixed correlators

Plot shows divergence of local current in each correlator,

$$\text{e.g., } \left[ \sum_x e^{iQ \cdot (x + \hat{\mu}/2)} \langle V_\mu^a(x) V_\nu^a(0) \rangle \right] \cdot \hat{Q}_\nu \neq 0$$

(Conserved DWF currents satisfy vector Ward identity & PCAC)



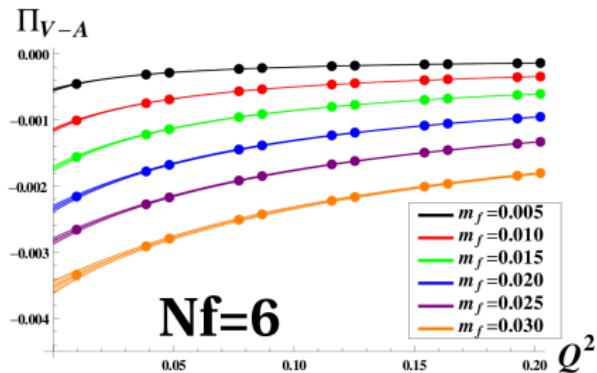
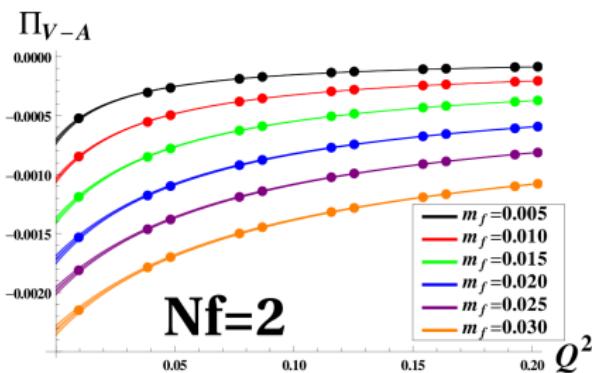
Cancellation seems due to conserved currents forming exact multiplet,  
also possible with overlap — even staggered (Y. Aoki @ Lattice 2013)

# Slope at $Q^2 = 0$ from fit to rational function

Very smooth data  $\Rightarrow$  fit to “Padé-(1, 2)” functional form (cf. Aubin et al.)

$$\Pi_{V-A}(Q^2) = \frac{a_0 + a_1 Q^2}{1 + b_1 Q^2 + b_2 Q^4} = \frac{\sum_{m=0}^1 a_m Q^{2m}}{\sum_{n=0}^2 b_n Q^{2n}}$$

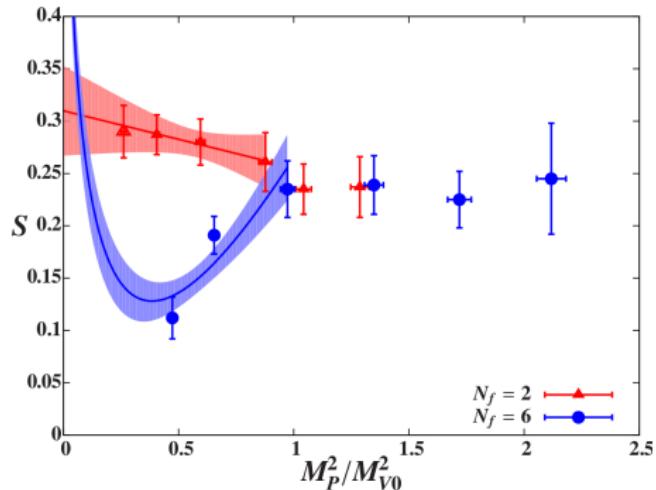
(motivations: single-pole dominance, Weinberg sum rules)



Results stable and  $\chi^2/dof \ll 1$  as  $Q^2$  fit range varies

Sannino, [arXiv:1006.0207](https://arxiv.org/abs/1006.0207): take  $m \rightarrow 0$  first in conformal window

# $S$ parameter, $N_F = 2$ and $N_F = 6$

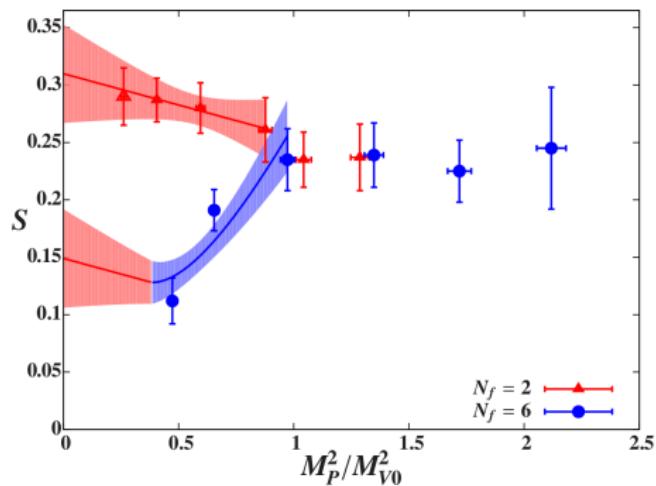


$$\lim_{M_P^2 \rightarrow 0} S = 0.31(4) \quad \text{for } N_F = 2$$

Linear fit to light points ( $M_P \leq M_{V0}$ ) guides the eye,  
accounts for any chiral logs remaining after  $\Delta S_{SM}$

$$S = A + B \frac{M_P^2}{M_{V0}^2} + \frac{1}{12\pi} \left( \frac{N_F}{2} - 1 \right) \log \left( \frac{M_{V0}^2}{M_P^2} \right) \text{ for } N_D = 1$$

# Phenomenologically-relevant chiral limit



$$\lim_{M_P^2 \rightarrow 0} S = 0.31(4)$$

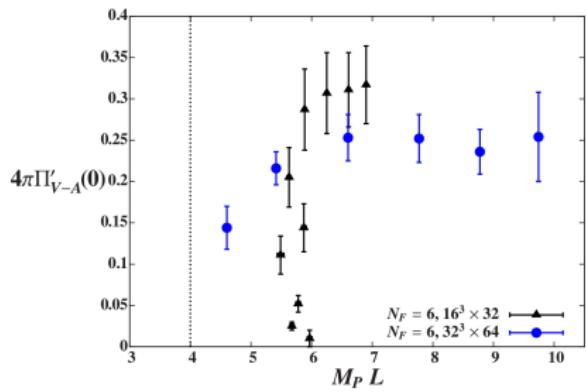
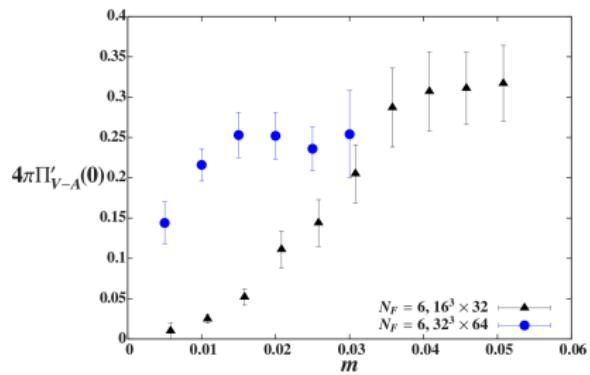
for  $N_F = 2$

- Lattice calculation involves  $N_F^2 - 1$  degenerate pseudoscalars
- Only three massless NGBs eaten in Higgs mechanism,  
 $N_F^2 - 4$  must be massive PNGBs

Imagine freezing  $N_F^2 - 4$  PNGB masses at the blue curve's minimum, and taking only three to zero mass

# Beware spurious $S \rightarrow 0$ from finite volume effects

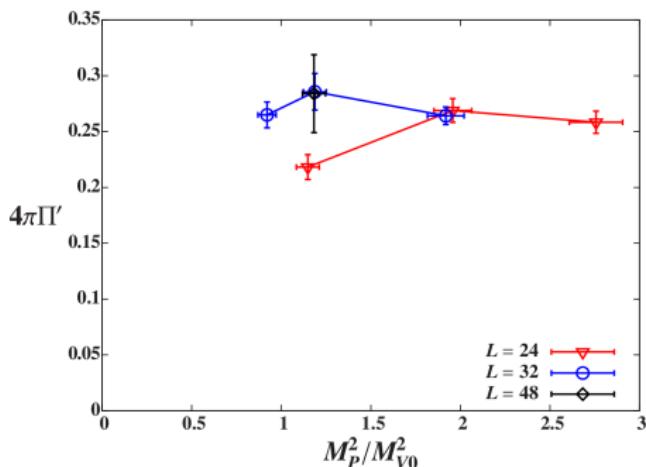
Compare  $N_F = 6$  results on  $16^3 \times 32$  vs.  $32^3 \times 64$   
 $L = 16$  results crash to zero as  $m \rightarrow 0$



As  $L = 16$  results crash to zero,  $M_P$  freezes around  $M_P L \approx 5.5$   
 $L = 32$  results show no such behavior,  
but we want more quantitative control over finite-volume effects

# Initial results for $N_F = 8$ domain wall on staggered

- Last week I spoke about the first USBSM project: [\(slides\)](#)  
 $N_F = 8$  staggered on larger volumes ( $48^3 \times 96$ ,  $64^3 \times 128$ )
- I am carrying out mixed-action measurements for  $S$
- May provide more information about finite-volume effects for  $N_F = 6$



Lightest  $24^3 \times 48$  point shows clear finite-volume effects

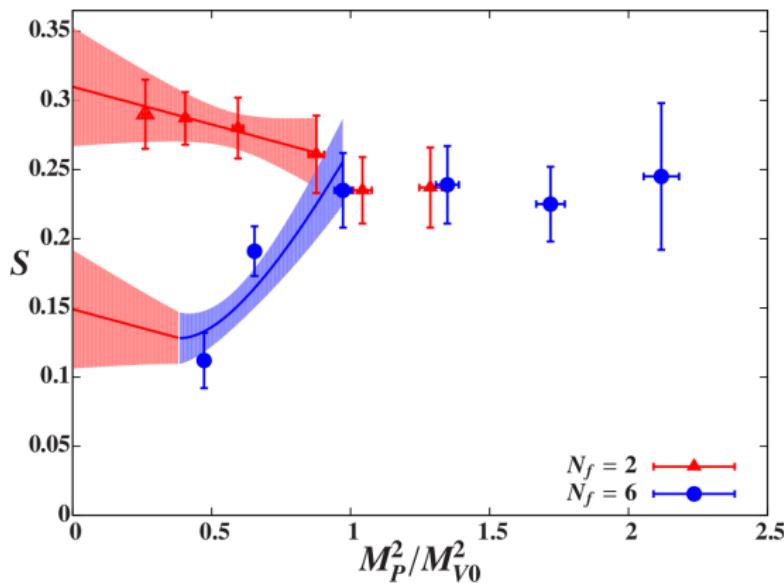
$N_F = 6$  reduction began only for  $M_P^2 \lesssim M_{V0}^2$

Also working on calculation with staggered fermions

Currently measuring smaller masses on  $48^3 \times 96$ , generating  $64^3 \times 128$

# Recapitulation

You saw this plot yesterday (from 2013 USQCD white paper)



Today I informally summarized how I made it  
Many backup slides follow in case there are questions

## Backup: Polarization functions for the $S$ parameter

$$\gamma \text{---} \bullet \text{---} \gamma = ig_2 g_1 \cos \theta_w \sin \theta_w \Pi_{ee} \delta^{\mu\nu} + \dots$$

$$Z \sim \text{wavy line} \quad \text{black circle} \quad \text{wavy line} \gamma = ig_2 g_1 (\Pi_{3e} - \sin^2 \theta_w \Pi_{ee}) \delta^{\mu\nu} + \dots$$

$$Z \sim \text{wavy line} \text{ (black circle)} \sim Z = \frac{i g_2 g_1}{\cos \theta_w \sin \theta_w} (\Pi_{33} - 2 \sin^2 \theta_w \Pi_{3e} + \sin^4 \theta_w \Pi_{ee}) \delta^{\mu\nu} + \dots$$

$$W \sim \text{wavy line} \sim W = ig_2^2 \Pi_{11} \delta^{\mu\nu} + \dots$$

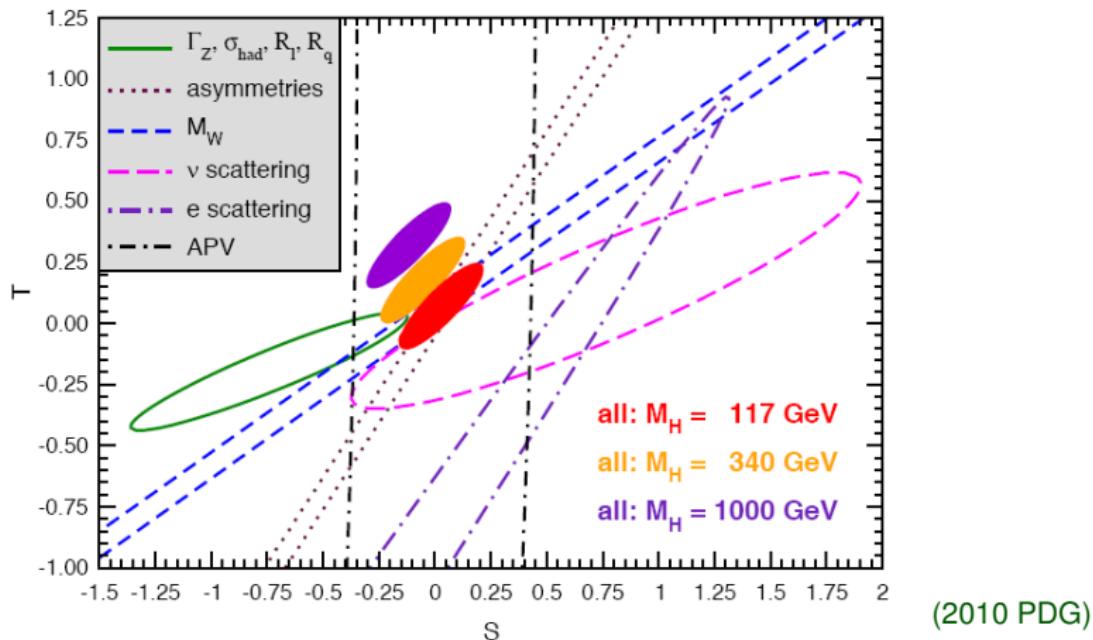
$$\Pi_{VV} = 2\Pi_{3e}$$

$$\Pi_{AA} = 4\Pi_{33} - 2\Pi_{3e}$$

$$S = 4\pi N_D \lim_{Q^2 \rightarrow 0} \frac{d}{dQ^2} \left[ \Pi_{VV}(Q^2) - \Pi_{AA}(Q^2) \right] - \Delta S_{SM}$$

# Backup: Experimentally, $S = 0.03(10)$ (Gfitter, arXiv:1209.2716)

For the old reference Higgs mass scale of 1 TeV,  $S \approx -0.15(10)$



- ▶  $Z$  decay partial widths and asymmetries
- ▶  $M_W, M_Z$
- ▶ Neutrino scattering cross sections
- ▶ Atomic parity violation

## Backup: Scaling up QCD gives $S \gtrsim 0.3$

$N_F \geq 2$  fermions in fundamental rep of  $SU(N)$  for  $N \geq 3$ ,  
**all**  $N_D = N_F/2$  doublets given chiral electroweak charges

$$S \simeq 0.3 \frac{N_F}{2} \frac{N}{3} + \frac{1}{12\pi} \left( \frac{N_F^2}{4} - 1 \right) \log \left( \frac{M_V^2}{M_P^2} \right)$$

- ① **Resonance contribution** uses QCD phenomenology to model  $R(s)$

$$4\pi \lim_{Q^2 \rightarrow 0} \frac{d}{dQ^2} \Pi_{V-A}(Q^2) = \frac{1}{3\pi} \int_0^\infty \frac{ds}{s} [R_V(s) - R_A(s)]$$

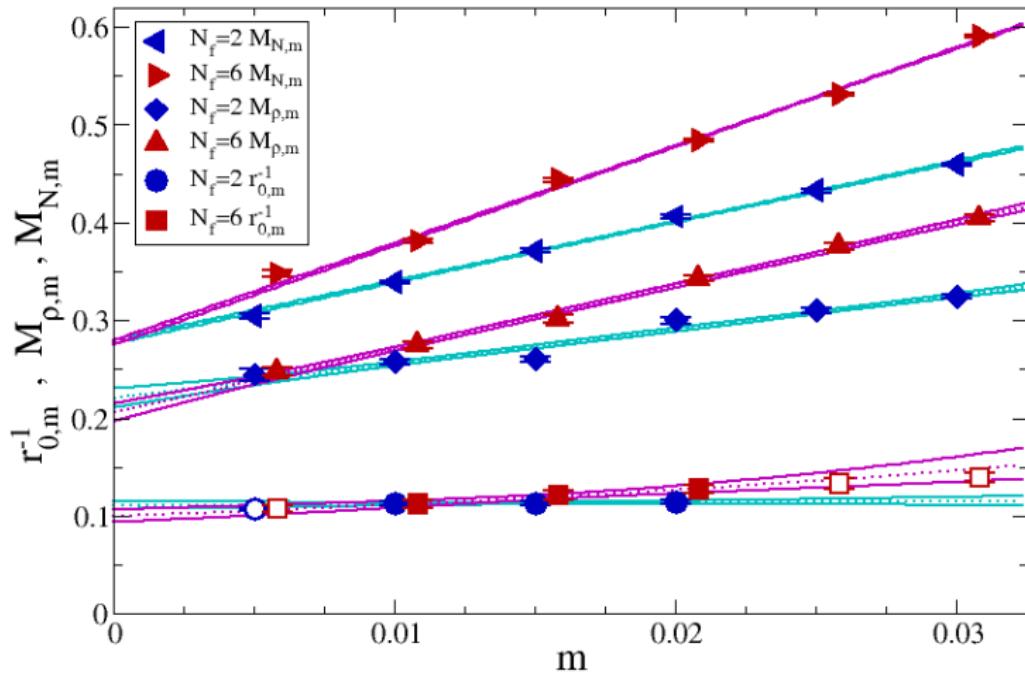
(essentially vector meson dominance with large- $N$  scaling)

- ② **Chiral-log contribution** based on leading-order chiral pert. theory

Both contributions are non-negative; first is  $\gtrsim 0.3$ , second  $\geq 0$

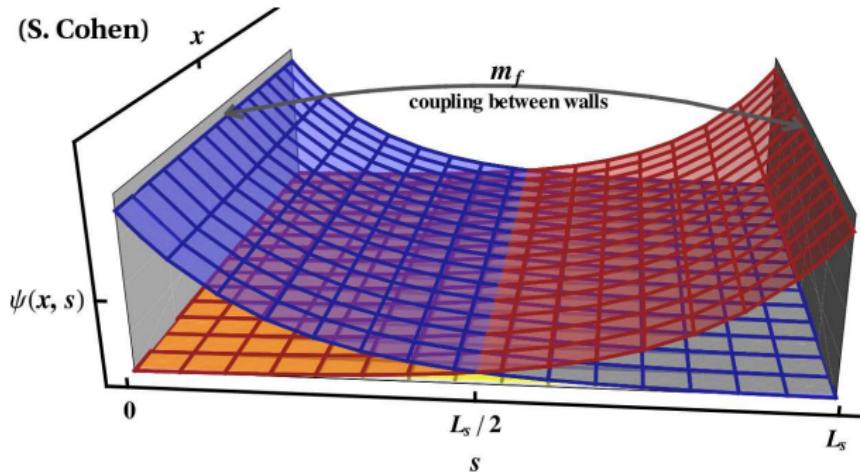
Can reduce many- $N_F$  logs by giving chiral charges to only one doublet

## Backup: Matching IR scales in the chiral limit



Vector mass, nucleon mass, and inverse Sommer scale  
all match at 10% level between  $N_F = 2$  and  $N_F = 6$   
 $M_{V0} = 0.215(3)$  [2f];  $0.209(3)$  [6f]

## Backup: More on domain wall fermions



- Domain wall fermions add fifth dimension of length  $L_s$ ,  
a significant computational expense
- Exact chiral symmetry at finite lattice spacing in the limit  $L_s \rightarrow \infty$ ,  
with nice continuum-like currents and flavor symmetries
- At finite  $L_s = 16$ , “residual mass”  $m_{\text{res}} \ll m_f$ ;  $m = m_f + m_{\text{res}}$

# Backup: Conserved and local domain wall currents

Conserved currents:

$$\mathcal{V}^{\mu a}(x) = \sum_{s=0}^{L_s-1} j^{\mu a}(x, s) \quad \mathcal{A}^{\mu a}(x) = \sum_{s=0}^{L_s-1} \text{sign}\left(s - \frac{L_s - 1}{2}\right) j^{\mu a}(x, s)$$

$$j^{\mu a}(x, s) = \bar{\Psi}(x + \hat{\mu}, s) \frac{1 + \gamma^\mu}{2} \tau^a U_{x,\mu}^\dagger \Psi(x, s) \\ - \bar{\Psi}(x, s) \frac{1 - \gamma^\mu}{2} \tau^a U_{x,\mu} \Psi(x + \hat{\mu}, s)$$

Local currents:

$$V^\mu(x) = \bar{q}(x) \gamma^\mu \tau^a q(x) \quad A^\mu(x) = \bar{q}(x) \gamma^\mu \gamma^5 \tau^a q(x)$$

$$q(x) = \frac{1 - \gamma^5}{2} \Psi(x, 0) + \frac{1 + \gamma^5}{2} \Psi(x, L_s - 1)$$

## Backup: More on current correlators

$$S = 4\pi N_D \lim_{Q^2 \rightarrow 0} \frac{d}{dQ^2} \Pi_{V-A}(Q^2) - \Delta S_{SM}$$



$$\Pi_{V-A}^{\mu\nu}(Q) = Z \sum_x e^{iQ \cdot (x + \hat{\mu}/2)} \text{Tr} \left[ \left\langle \mathcal{V}^{\mu a}(x) V^{\nu b}(0) \right\rangle - \left\langle \mathcal{A}^{\mu a}(x) A^{\nu b}(0) \right\rangle \right]$$

$$\Pi^{\mu\nu}(Q) = \left( \delta^{\mu\nu} - \frac{\hat{Q}^\mu \hat{Q}^\nu}{\hat{Q}^2} \right) \Pi(Q^2) - \frac{\hat{Q}^\mu \hat{Q}^\nu}{\hat{Q}^2} \Pi^L(Q^2) \quad \hat{Q} = 2 \sin(Q/2)$$

- Renormalization constant  $Z = Z_A = Z_V$  for chiral fermions  
Computed non-perturbatively:  $Z = 0.85$  (2f);  $0.73$  (6f)
- Conserved currents  $\mathcal{V}$  and  $\mathcal{A}$  ensure that lattice artifacts cancel

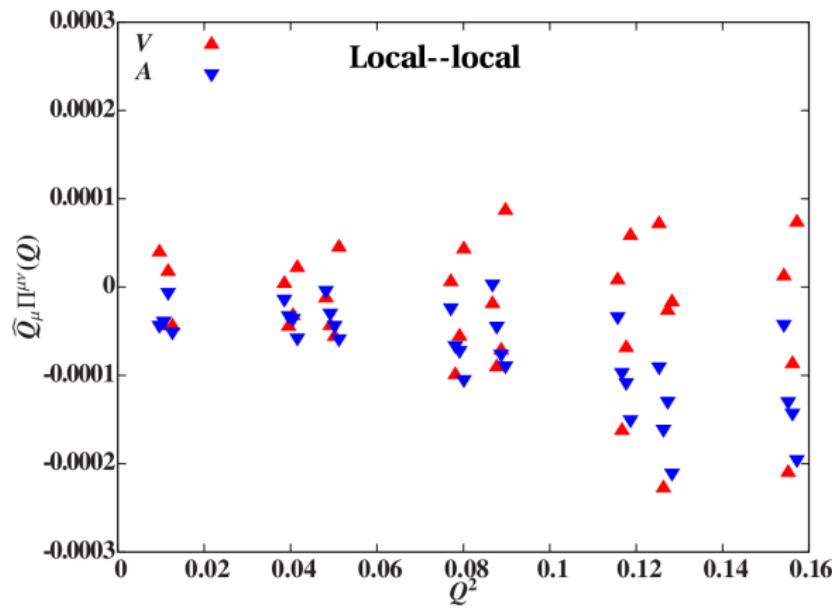
## Backup: More on local domain wall currents

$$\Pi_{V-A}^{\mu\nu}(Q) = Z \sum_x e^{iQ \cdot (x + \hat{\mu}/2)} \text{Tr} \left[ \left\langle V^{\mu a}(x) V^{\nu b}(0) \right\rangle - \left\langle A^{\mu a}(x) A^{\nu b}(0) \right\rangle \right]$$

“Local” currents are simple  $\bar{q}\gamma^\mu q$ , defined on domain walls

No Ward identity:

$$\hat{Q}_\mu \left[ \sum_x e^{iQ \cdot x} \langle V_\mu^a(x) V_\nu^a(0) \rangle \right] \neq 0$$

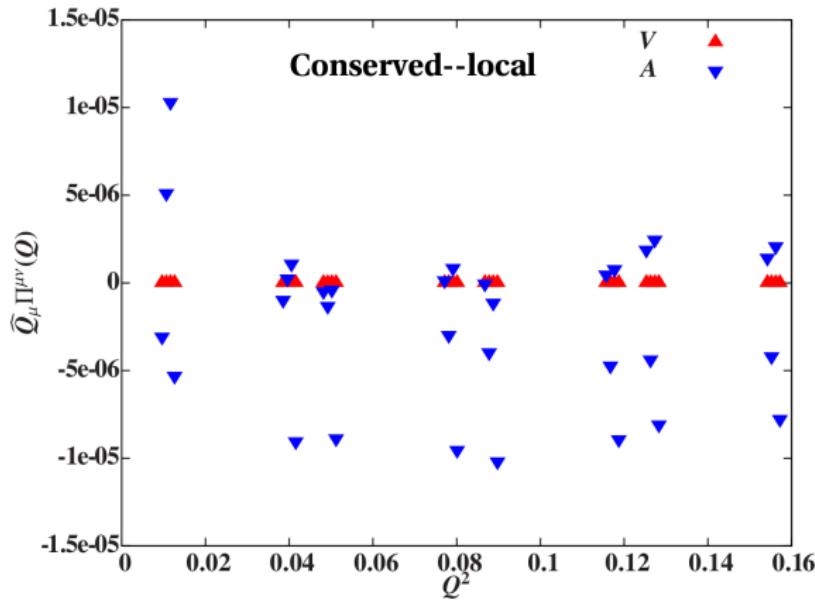


## Backup: More on conserved domain wall currents

$$\Pi_{V-A}^{\mu\nu}(Q) = Z \sum_x e^{iQ \cdot (x + \hat{\mu}/2)} \text{Tr} \left[ \left\langle \mathcal{V}^{\mu a}(x) V^{\nu b}(0) \right\rangle - \left\langle \mathcal{A}^{\mu a}(x) A^{\nu b}(0) \right\rangle \right]$$

Conserved currents are point-split, summed over fifth dimension

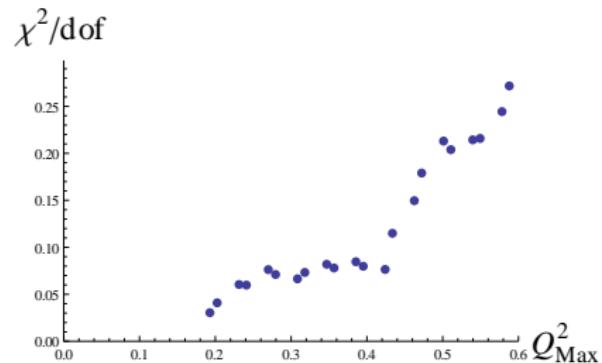
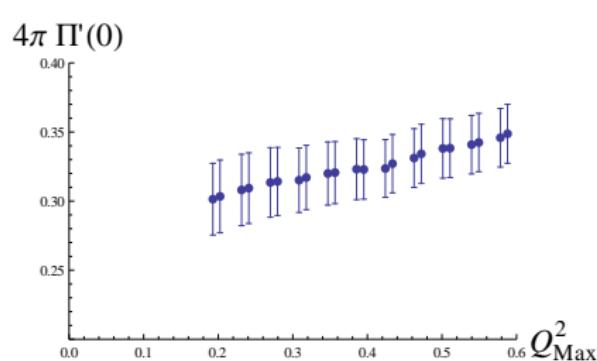
Obey Ward identity, PCAC:  $\hat{Q}_\mu [\sum_x e^{iQ \cdot (x + \hat{\mu}/2)} \langle \mathcal{V}_\mu^a(x) V_\nu^a(0) \rangle] = 0$



# Backup: Padé fit $Q^2$ -range dependence

Fitting to “Padé-(1, 2)” rational function,

$$\Pi_{V-A}(Q^2) = \frac{a_0 + a_1 Q^2}{1 + b_1 Q^2 + b_2 Q^4} = \frac{\sum_{m=0}^1 a_m Q^{2m}}{\sum_{n=0}^2 b_n Q^{2n}}$$

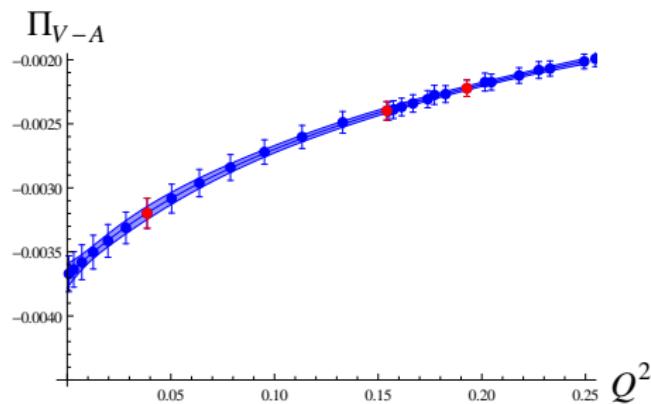


Results reported here used  $Q_{\text{Max}} = 0.4$

# Backup: Twisted boundary conditions for $\Pi_{V-A}(Q^2)$

## Twisted boundary conditions (TwBCs)

- Introduce external abelian field (add phase at lattice boundaries)
- Allows access to arbitrary  $Q^2$ , not just lattice modes  $2\pi n/L$



- Correlations  $\Rightarrow$  TwBCs do not improve Padé fit results for slope
- May help in conformal window,  
where we want to extrapolate  $m \rightarrow 0$  at small fixed  $Q^2$

# Backup: Chiral perturbation theory for $\Pi_{V-A}(Q^2)$

(Used by JLQCD & RBC–UKQCD in QCD projects)

- Effective field theory predicting dependence on  $M_P^2$  and  $Q^2$
- Need both  $M_P^2$  and  $Q^2$  smaller than ours
- $N_F > 2$  produces complications

For  $N_F = 2$ ,

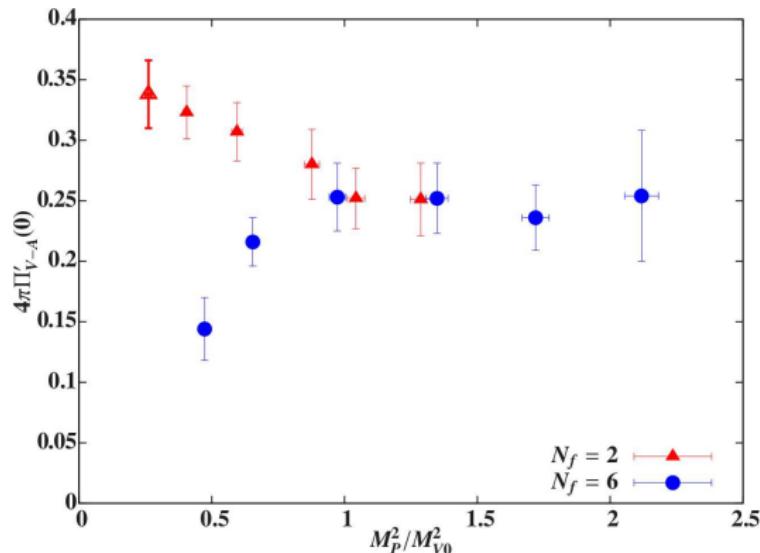
(Gasser and Leutwyler, 1984)

$$S = \frac{1}{12\pi} \left( \bar{\ell}_5 + \log \left[ \frac{M_P^2 \frac{v^2}{F_P^2}}{M_H^2} \right] - \frac{1}{6} \right)$$

$$\Pi_{V-A}(Q^2) = -F_P^2 + Q^2 \left[ \frac{1}{24\pi^2} \left( \bar{\ell}_5 - \frac{1}{3} \right) + \frac{2}{3}(1+x)\bar{J}(x) \right]$$

$$\bar{J}(x) = \frac{1}{16\pi^2} \left( \sqrt{1+x} \log \left[ \frac{\sqrt{1+x}-1}{\sqrt{1+x}+1} \right] + 2 \right), \quad x \equiv 4M_P^2/Q^2$$

## Backup: Fit results for $\Pi'_{V-A}(0)$ , $N_F = 2$ and $N_F = 6$



Vertical axis:  $4\pi\Pi'_{V-A}(0)$

where

$$\Pi'(0) = \lim_{Q^2 \rightarrow 0} \frac{d}{dQ^2} \Pi(Q^2)$$

$$S = 4\pi N_D \Pi'_{V-A}(0) - \Delta S_{SM}$$

Horizontal axis:  $M_P^2/M_{V0}^2$  gives a more physical comparison than  $m_f$

$M_{V0} \equiv \lim_{m \rightarrow 0} M_V$  is matched between  $N_F = 2$  and  $N_F = 6$

Expect agreement in the quenched limit  $M_P^2 \rightarrow \infty$

## Backup: From slopes to $S$

$$S = 4\pi N_D \lim_{Q^2 \rightarrow 0} \frac{d}{dQ^2} \Pi_{V-A}(Q^2) - \Delta S_{SM}$$

- ①  $N_D$  doublets with **chiral** electroweak couplings contribute to  $S$   
Scaled-up QCD usually considers **maximum**  $N_D = N_F/2$   
but only  $N_D \geq 1$  is required for electroweak symmetry breaking

$$\textcircled{2} \quad \Delta S_{SM} = \frac{1}{4} \int_{4M_P^2}^{\infty} \frac{ds}{s} \left[ 1 - \left( 1 - \frac{M_{V0}^2}{s} \right)^3 \Theta(s - M_{V0}^2) \right]$$

No direct dependence on  $N_F$  or  $N_D$

Diverges logarithmically in the limit  $M_P^2 \rightarrow 0$

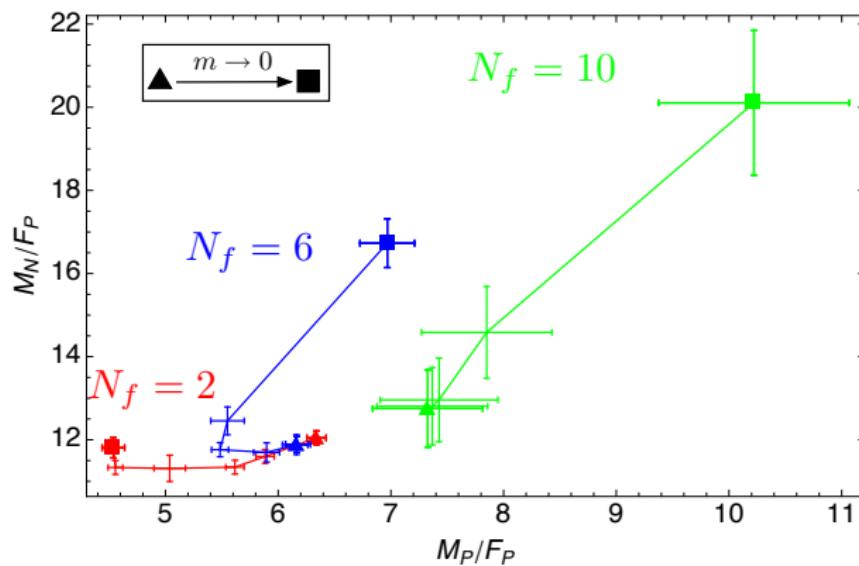
to cancel contribution of three eaten modes

Small ( $\lesssim 10\%$ ) reduction in this work ( $M_P^2 > 0$ )

## Backup: More on finite-volume effects

Range of accessible masses determined by lattice volume

If masses get too small, finite-volume effects significant



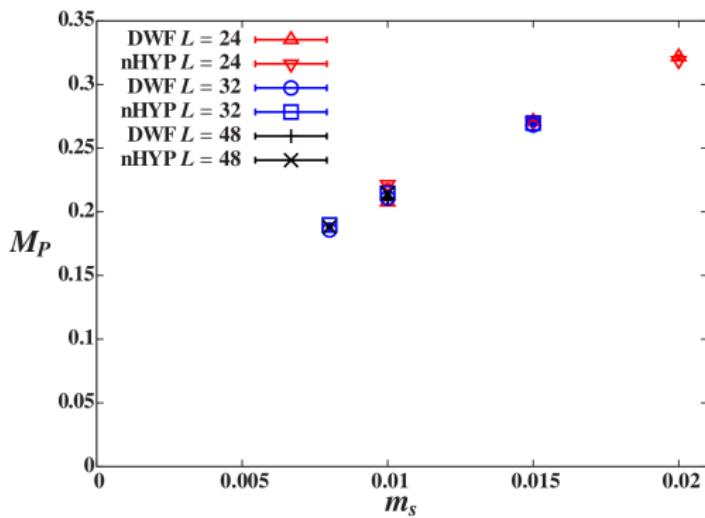
In this diagnostic, finite-volume effects push points up and to the right by increasing the masses but decreasing  $F_P$

# Backup: Mixed action procedure (LHPC, arXiv:0705.4295)

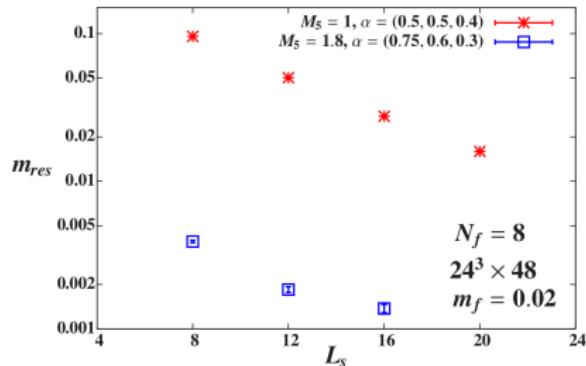
- HYP smear to reduce  $m_{res}$  and get renormalization factors  $Z \sim 1$
- Tune domain wall height  $M_5$  and length  $L_s$  of fifth direction so that residual chiral symmetry breaking  $m_{res} \ll m$
- Tune bare valence mass  $m_f$  so that  $M_P$  matches unitary value

$M_5 = 1.8$  and  $L_s = 16$   
 $\rightarrow Z_V \approx Z_A \approx 1.08$   
 $\rightarrow m_{res} \approx 0.001 \lesssim m_f/13$

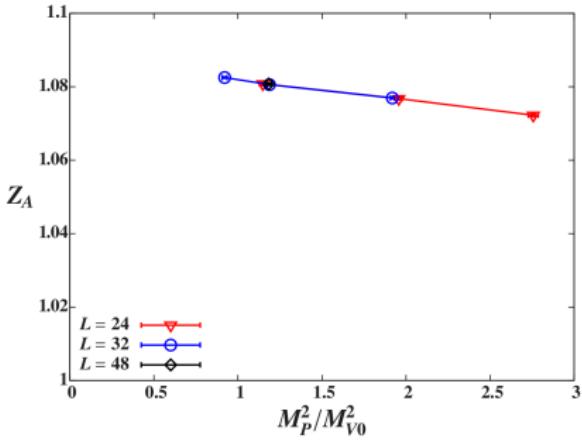
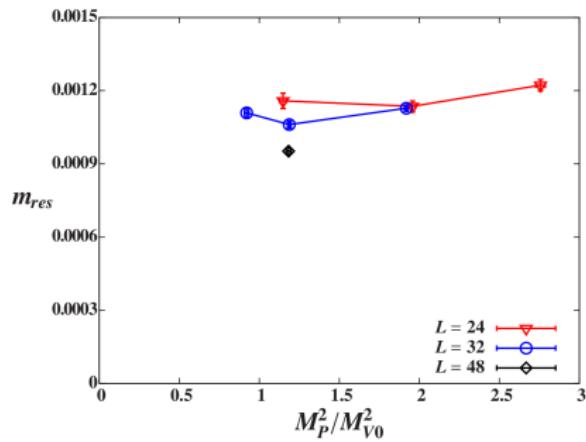
Need  $m_f > m_s$  to match  $M_P$ :  
 $1.7 \lesssim m/m_s \lesssim 2.05$   
where  $m \equiv m_f + m_{res}$



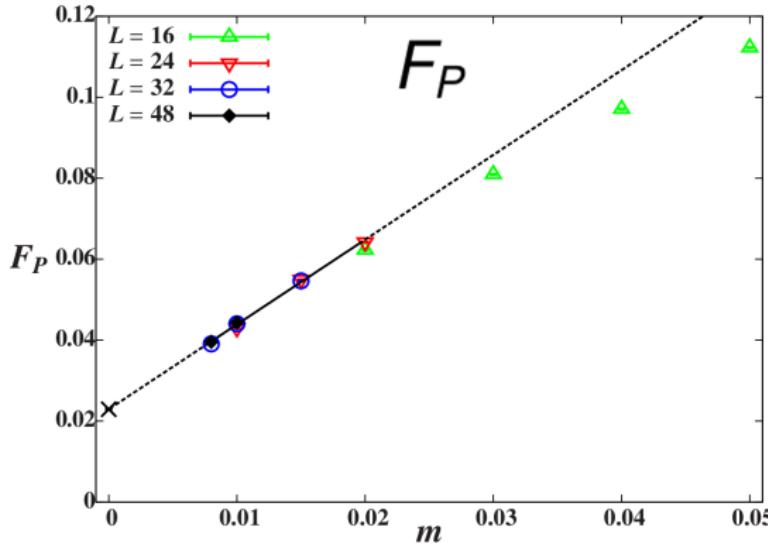
# Backup: Valence domain wall $m_{res}$ and $Z_A$



- Want  $m_{res} \ll m$
- Want  $Z_A \sim 1$



## Backup: pseudoscalar decay constant



For  $N_F = 8$  staggered,  
 $1 \leq F_P L \leq 2.1$

Fitted points:  $1.25 \leq F_P L \leq 1.6$

Intercept  $0.023 \approx 1.1/48$

For  $N_F = 2$  domain wall,  $0.8 \leq F_P L \leq 1.2$ , intercept  $0.026 \approx 0.8/32$

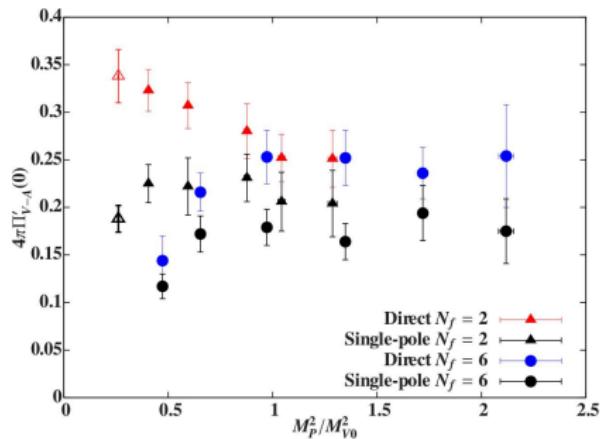
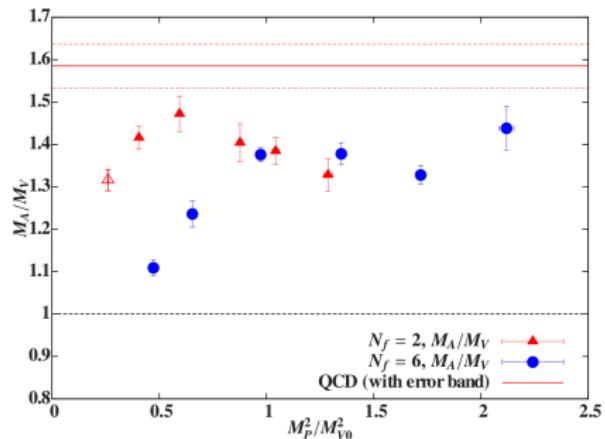
For  $N_F = 6$  domain wall,  $0.7 \leq F_P L \leq 1.6$ , intercept  $0.019 \approx 0.6/32$

All simple linear extrapolations – NLO $\chi$ PT may have significant effect

# Backup: Connection to parity-doubling

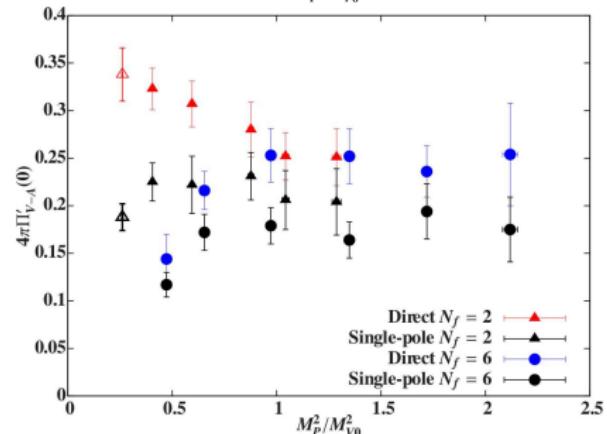
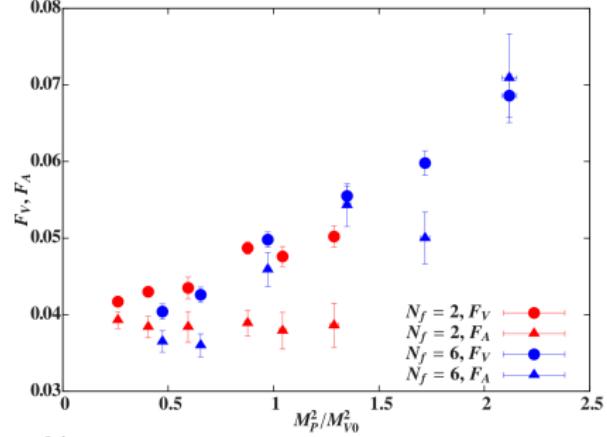
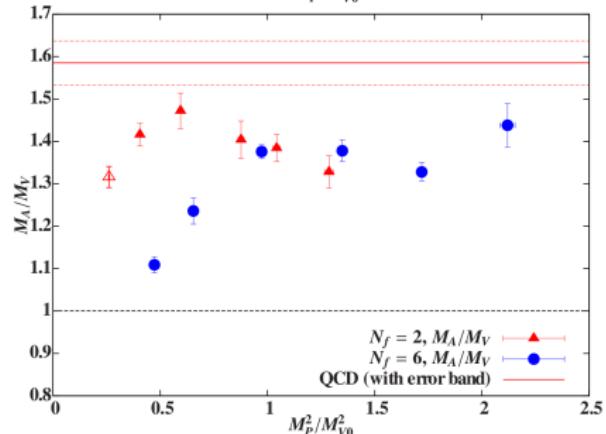
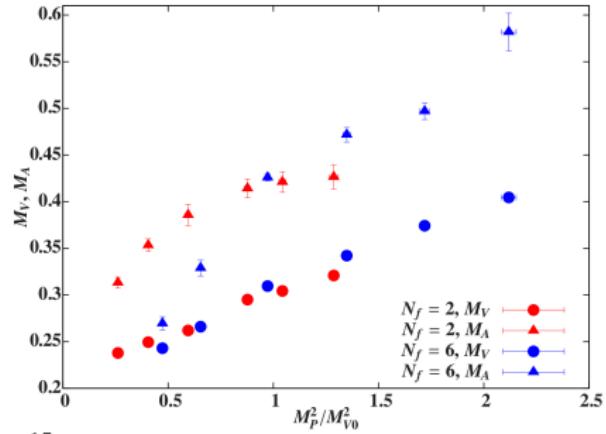
$$4\pi\Pi'_{V-A}(0) = \frac{1}{3\pi} \int_0^\infty \frac{ds}{s} [R_V(s) - R_A(s)] \longrightarrow 4\pi \left[ \frac{F_V^2}{M_V^2} - \frac{F_A^2}{M_A^2} \right]$$

for  $R(s) \longrightarrow 12\pi F^2 \delta(s - M^2)$



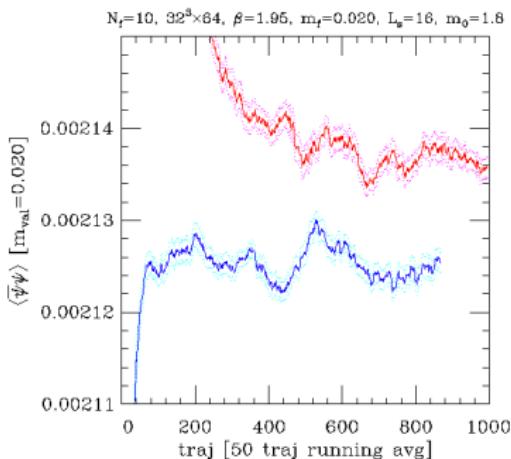
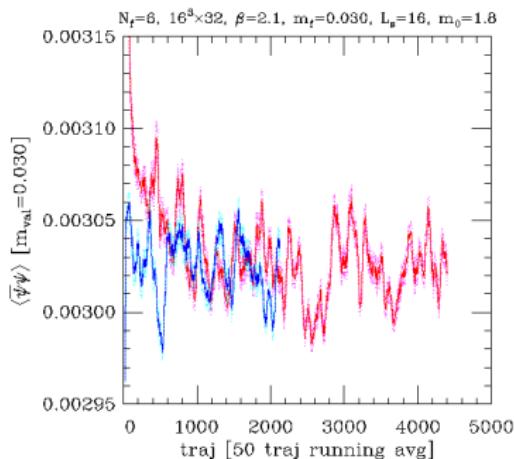
Signs of  $N_F = 6$  parity-doubling consistent with reduced  $S$   
Direct checks of finite-volume effects underway

# Backup: More parity-doubling plots



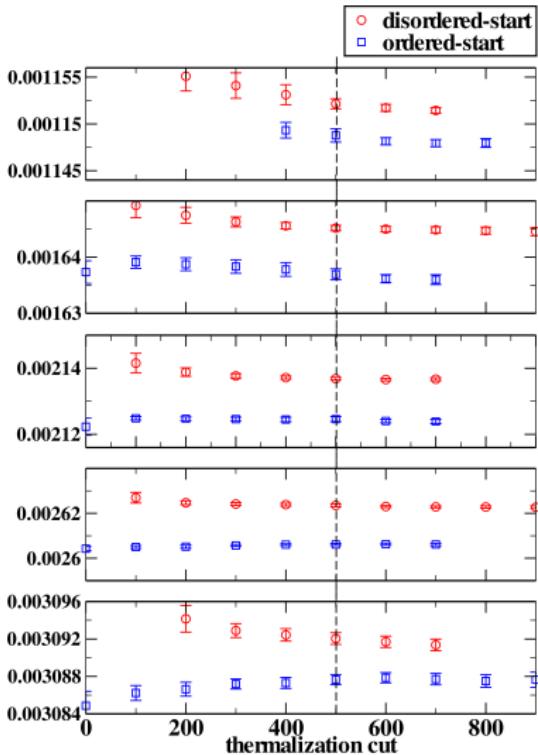
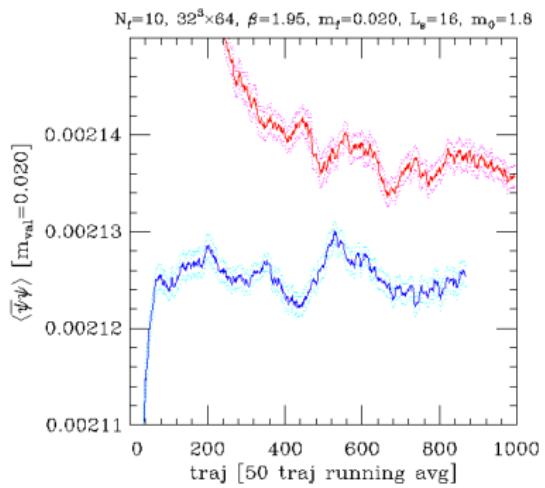
# Backup: $N_F = 10$ thermalization and autocorrelations

- We generate a Markov chain of gauge field configurations  
     $\Rightarrow$  Nearby links in the chain are correlated
- From initial state, system thermalizes to equilibrium distribution
- Independent measurements require autocorrelations to die off



Independent ensembles starting from either random or ordered states  
 $\Rightarrow$  Find thermalization time from convergence to equilibrium

# Backup: $N_F = 10$ thermalization and autocorrelations



- Random and ordered initial states
- Goal: convergence to equilibrium  
     $\Rightarrow$  thermalization time
- Result: two independent samples