Lattice supersymmetry in a nutshell

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Goals of this informal and pedagogical introduction

- Write down supersymmetric lattice system
  (Four-dimensional $\mathcal{N} = 4$ SYM — analogous lattice systems in 2d & 3d)
- Point out connections to staggered fermions
- Raise issue of potential sign problem,
  possible connections to complex Langevin and Lefschetz thimble methods
- **Main reference:** arXiv:0903.4881
- **Skip motivations:** Take it for granted that we care about lattice susy

Supersymmetry and naive lattice obstacle

- **Supersymmetries extend Poincaré spacetime symmetry**
- **Lorentz:** Work in euclidean space $\longrightarrow$ SO$(d)_{\text{euc}}$ rotations $\Lambda_{\mu\nu}$
  "Time" arbitrary – transfer matrix can be defined along any lattice vector
- **Poincaré:** Add spacetime translations P_μ to (euclidean) Lorentz

$$[P_\mu, P_\nu] = 0 \qquad [P_\mu, \Lambda_{\rho\sigma}] \propto \delta_{\mu\rho} P_\sigma - \delta_{\mu\sigma} P_\rho$$
$$[\Lambda_{\mu\nu}, \Lambda_{\rho\sigma}] \propto \delta_{\mu\rho}\Lambda_{\nu\sigma} + \delta_{\nu\sigma}\Lambda_{\mu\rho} - \delta_{\mu\sigma}\Lambda_{\nu\rho} - \delta_{\nu\rho}\Lambda_{\mu\sigma}$$

- **Supercharges:** Spinorial generators Q_α^I and $\overline{Q}_{\dot\alpha}^I$ with $I = 1, \cdots, \mathcal{N}$
  Transform under global SU$(\mathcal{N})_R$ flavor symmetry ("R symmetry")

$$\left\{Q_\alpha^I, Q_\beta^J\right\} = 0 \qquad \left\{Q_\alpha^I, \overline{Q}_{\dot\alpha}^J\right\} = 2\delta^{IJ}\sigma^\mu_{\alpha\dot\alpha} P_\mu$$
$$\left[Q_\alpha^I, P_\mu\right] = 0 \qquad \left[Q_\alpha^I, \Lambda_{\mu\nu}\right] \propto \frac{1}{4}\left[\gamma_\mu, \gamma_\nu\right]_\alpha^\beta Q_\beta^I$$

- **Lattice:** P_μ generates infinitesimal spacetime translations
  P_μ does not exist in discrete spacetime $\Longrightarrow$ no supersymmetry protection,
  have to fine tune (typically many) relevant or marginal operators
- **Aside:** Banks & Windey '82 tried using hamiltonian formulation
  to preserve $\left\{Q_\alpha^I, \overline{Q}_{\dot\alpha}^J\right\} = 2\delta^{IJ}\sigma^0_{\alpha\dot\alpha} H$
  Now fine-tuning required to recover Lorentz symmetry in continuum limit

Kähler–Dirac selects $\mathcal{N} = 4$ SYM in 4d

- **Consider 4d:** Expand 4×4 matrix of 16 supercharges in basis of γ matrices

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

- **Observation:** Simple change of variables (in flat spacetime) that replaces spinors with anti-symmetric tensors

- **Key:** Susy subalgebra $\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$ can be preserved on lattice

- **Restriction:** Need at least 2^d supercharges for expansion
Only one possibility in 4d: $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM)

- $\mathcal{N} = 4$ **SYM:** Restricting to spins ≤ 1 allows only single super-multiplet
Contains the gauge field A_μ, four fermions Ψ^I and six scalars Φ^{IJ} all in adoint rep. of gauge group — typically SU(N)

State	Helicity	Flavor $SU(4)_R$
$\lvert\Omega_1\rangle$	1	**1**
$Q^I_\alpha \lvert\Omega_1\rangle$	1/2	**4**
$Q^J_\beta\, Q^I_\alpha \lvert\Omega_1\rangle$	0	**6**
$Q^K_\gamma\, Q^J_\beta\, Q^I_\alpha \lvert\Omega_1\rangle$	−1/2	$\overline{\mathbf{4}}$
$Q^L_\delta\, Q^K_\gamma\, Q^J_\beta\, Q^I_\alpha \lvert\Omega_1\rangle$	−1	**1**

- **Conformal:** $\beta = 0$ for all couplings (line of fixed points)

Connection to (reduced) staggered fermions

- **Four fermions:** Majorana Ψ^I expand just like supercharges

$$\begin{pmatrix} \Psi^1 & \Psi^2 & \Psi^3 & \Psi^4 \end{pmatrix} \longrightarrow (\eta, \psi_\mu, \chi_{\mu\nu}, \overline{\psi}_\mu, \overline{\eta})$$

- **Observation:** Expansion mixes flavor symmetry (horizontal in matrix) and spacetime symmetry (vertical in matrix)
Equivalent to (reduced) staggered fermions: Banks, Dothan & Horn '82
More formal Kähler–Dirac foundation in Rabin '82, Becher & Joos '82

Topological twisting

- **More formally:** Matrix gives $\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R$
  We're expanding in reps of "twisted rotation group"
  $$\text{SO}(4)_{tw} \equiv \text{diag}\Big[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R\Big]$$
- **Complication:** Only have $\text{SO}(4)_R \subset \text{SU}(4)_R \simeq \text{SO}(6)_R$
  $\Longrightarrow$ Scalar fields $\Phi^{IJ} \longrightarrow (B_\mu, \phi, \overline{\phi})$
- **Solution:** Combine $4+6$ bosons in complexified $\mathcal{A}_a = (A_\mu, \phi) + i(B_\mu, \overline{\phi})$
  Similarly combine $\psi_a = (\psi_\mu, \overline{\eta})$ and $\chi_{ab} = (\chi_{\mu\nu}, \overline{\psi}_\mu)$
- **Question:** Could complexified gauge field be related to (hep-lat/0301028)
  complex Langevin or Lefschetz thimble approaches to sign problem?

A_4^* lattice and its S_5 point group symmetry

- **A_4^*:** Contains five links symmetrically spanning four dimensions
  Four-dimensional analog of 2d triangular lattice
  Can obtain from dimensional reduction with symmetric constraint $\sum_a \partial_a = 0$
- **S_5 point group symmetry:** S_5 irreps match those of $\text{SO}(4)_{tw}$
  Extracted by orthogonal matrix
  $$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{3}{\sqrt{12}} & 0 \\ \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & -\frac{4}{\sqrt{20}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}.$$
  $$\Longrightarrow \begin{pmatrix} \psi_\mu \\ \\ \overline{\eta} \end{pmatrix} = P \begin{pmatrix} \psi_a \end{pmatrix} \qquad \begin{pmatrix} \chi_{\mu\nu} \\ \\ \overline{\psi}_\mu \end{pmatrix} = PP \begin{pmatrix} \chi_{ab} \end{pmatrix}$$
- **Scalar $\mathcal{Q}$:** Nilpontent ($\mathcal{Q}^2 = 0$), exchanges bosons $\longleftrightarrow$ fermions
  $$\begin{aligned} &\mathcal{Q}\,\mathcal{U}_a = \psi_a && \mathcal{Q}\,\psi_a = 0 \\ &\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} && \mathcal{Q}\,\overline{\mathcal{U}}_a = 0 \\ &\mathcal{Q}\,\eta = d && \mathcal{Q}\,d = 0 \end{aligned}$$
  d is bosonic auxiliary field, with standard e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{U}_a$

Sign problem

- **Phase reweighting:** Allows importance sampling Monte Carlo using real non-negative Boltzmann factor $|\text{pf}\,\mathcal{D}|e^{-S_B}$

$$\begin{aligned}\langle\mathcal{O}\rangle &= \frac{1}{Z}\int[d\mathcal{U}_a][d\overline{\mathcal{U}}_a][d\Psi]\mathcal{O}e^{-S_B[\mathcal{U}_a,\overline{\mathcal{U}}_a]-\Psi^T\mathcal{D}[\mathcal{U}_a,\overline{\mathcal{U}}_a]\Psi} \\ &= \frac{1}{Z}\int[d\mathcal{U}_a][d\overline{\mathcal{U}}_a]\mathcal{O}e^{i\alpha}|\text{pf}\,\mathcal{D}|e^{-S_B} \\ &= \frac{\left\langle\mathcal{O}e^{i\alpha}\right\rangle_{pq}}{\left\langle e^{i\alpha}\right\rangle_{pq}}\end{aligned}$$

- **Sign problem:** When the phase α fluctuates so much that $\left\langle e^{i\alpha}\right\rangle_{pq}$ is consistent with zero

- **Numerical results:** Phase fluctuations strangely sensitive to temporal BCs
  $e^{i\alpha}\approx 1$ with anti-periodic BCs, $\left\langle e^{i\alpha}\right\rangle_{pq}\approx 0$ with periodic BCs
  Even more strangely, other observables change little for different BCs

Lattice action

- **Twisted action:** S is manifestly $\mathcal{Q}$-supersymmetric

$$S = \frac{N}{2\lambda_{\text{lat}}}\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab}+\eta\overline{\mathcal{D}}_a\mathcal{A}_a-\frac{1}{2}\eta d\right)-\frac{N}{8\lambda_{\text{lat}}}\epsilon_{abcde}\ \chi_{ab}\overline{\mathcal{D}}_c\chi_{de}$$

  $\mathcal{Q}S=0$ follows from $\mathcal{Q}^2\cdot=0$ and Bianchi identity $\epsilon_{abcde}\ \overline{\mathcal{D}}_c\overline{\mathcal{F}}_{de}=0$

- **Expand:** Apply $\mathcal{Q}$ and integrate out auxiliary field d:

$$\begin{aligned}S = \frac{N}{2\lambda_{\text{lat}}}&\left[-\overline{\mathcal{F}}_{ab}\mathcal{F}_{ab}+\frac{1}{2}\left(\overline{\mathcal{D}}_a\mathcal{U}_a\right)^2-\chi_{ab}\mathcal{D}_{[a}\psi_{b]}-\eta\overline{\mathcal{D}}_a\psi_a\right] \\ &-\frac{N}{8\lambda_{\text{lat}}}\epsilon_{abcde}\ \chi_{ab}\overline{\mathcal{D}}_c\chi_{de}\end{aligned}$$

Numerical complications

- **Non-compact links live in algebra:** $\mathcal{Q}\,\mathcal{U}_a = \psi_a \Longrightarrow \mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C})$
  Flat measure in path integral is gauge invariant due to complexification
  Need $\mathcal{U}_a = \frac{1}{a}\mathbb{I}_N + \mathcal{A}_a + \mathcal{O}(a^2)$ in continuum limit,
  stabilized by scalar potential $\sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2$

- **U(N) gauge invariance:** Due to complexified links
  $\text{U}(N) = \text{SU}(N) \otimes \text{U}(1)$ but U(1) only decouples in continuum

- **Flat directions:** Those in U(1) sector seem especially problematic
  Include all constant U(1) shifts of x-independent fields, even if $S \neq 0$
  SU(N) flat directions restricted to supersymmeric vacua with $S = 0$

- **Lifting:** Scalar potential lifts SU(N) flat directions but softly breaks susy
  Plaquette determinant lifts U(1) flat directions,
  can be implemented supersymmetrically (1505.03135)
  Modify e.o.m. for auxiliary field $d = \overline{\mathcal{D}}_a\mathcal{U}_a + 2G\text{Re}\ \sum_{a<b}\left(\det \mathcal{P}_{ab} - 1\right)\mathbb{I}_N$

PS: Motivations / context for lattice supersymmetry

- **BSM:** Supersymmetry most familiar as ingredient in new physics models
  Relies on (dynamical) spontaneous supersymmetry breaking $\longrightarrow$ lattice

- **Symmetries:** Simplify analytic calculations, allowing insight into
  confinement, dynamical symmetry breaking, conformality, ...
  Lattice is new non-perturbative method to confirm / refine / extend insights

- **Dualities:** As in spin systems (e.g., Kramers & Wannier on 2d Ising),
  theories with different fields & interactions produce identical physics
  Relate "electric" & "magnetic" gauge theories — Seiberg duality
  Relate gauge & gravity theories — AdS/CFT duality or "holography"
  Method: Conjecture & check (exploiting susy), may be extended by lattice

- **Modelling:** Attempts to study everything from QCD at finite density
  to non-Fermi liquids based on AdS/CFT holography
  Lattice could provide new input to these efforts — validate or refine