Supersymmetric Yang–Mills theories on the lattice

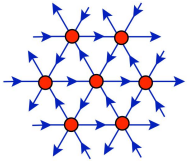
David Schaich (Liverpool)

Imperial College String Seminar, 27 October 2021

arXiv:1810.09282 arXiv:2010.00026 arXiv:2109.01001

and more to come with R. G. Jha, A. Joseph, A. Sherletov & T. Wiseman

Overview and plan

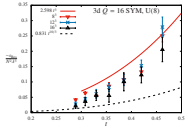


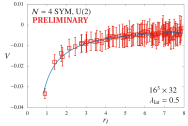
Preserve (some) susy in discrete space-time

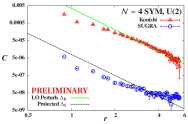
$\longrightarrow$ practical lattice investigations

First reproduce perturbative and holographic results,

then access new domains





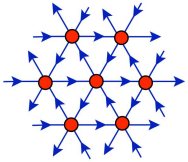


Why: Lattice supersymmetry motivation

How: Lattice $\mathcal{N} = 4$ SYM formulation highlights

What: Recent, ongoing & planned work

Overview and plan



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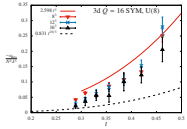
$\longrightarrow$ practical lattice investigations



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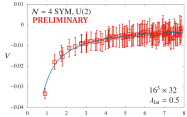
How: Lattice $\mathcal{N} = 4$ SYM formulation highlights

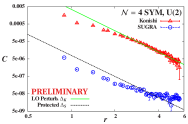


What: Recent, ongoing & planned work

Thermodynamics in 1+1 and 2+1 dimensions

(3+1)d static potential and scaling dimensions

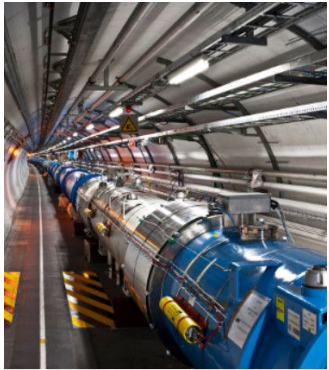
Sign problems, supersymmetric QCD, . . .



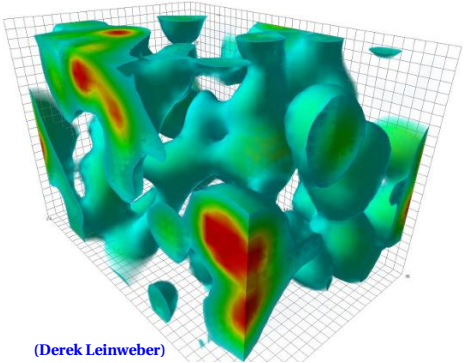
Motivations

Lattice field theory promises first-principles predictions
for strongly coupled supersymmetric QFTs

BSM

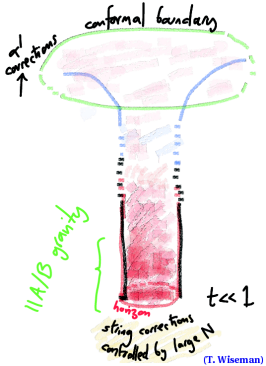


QFT



(Derek Leinweber)

Holography

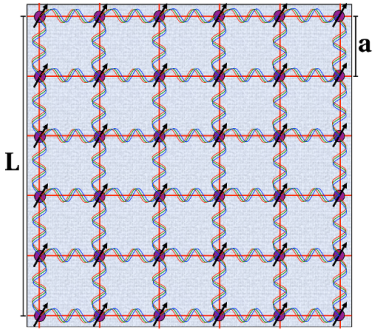


(T. Wiseman)

Lattice field theory in a nutshell

Formally $\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \;\; \mathcal{O}(\Phi) \;\; e^{-S[\Phi]}$

Regularize by formulating theory in finite, discrete, euclidean space-time
↖Gauge invariant, non-perturbative, d-dimensional



P. Vranas LLNL

Spacing between lattice sites ("a")
$\longrightarrow$ UV cutoff scale $1/a$

Remove cutoff: $a \to 0 \quad (L/a \to \infty)$

Discrete $\longrightarrow$ continuous symmetries ✓

Numerical lattice field theory calculations

High-performance computing $\longrightarrow$ evaluate up to $\sim$billion-dimensional integrals
(Dirac operator as $\sim 10^9 \times 10^9$ matrix)

Results to be shown, and work in progress, require state-of-the-art resources

Many thanks to USQCD–DOE, DiRAC–STFC–UKRI, and computing centres!

USQCD @Fermilab

DiRAC @Cambridge

Barkla @Liverpool

Numerical lattice field theory calculations

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Importance sampling Monte Carlo

Algorithms sample field configurations with probability $\frac{1}{\mathcal{Z}} e^{-S[\Phi]}$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \;\; \mathcal{O}(\Phi) \;\; e^{-S[\Phi]} \longrightarrow \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(\Phi_i) \;\; \text{with stat. uncertainty} \propto \frac{1}{\sqrt{N}}$$

Supersymmetry must be broken on the lattice

Supersymmetry is a space-time symmetry, $(\mathrm{I} = 1, \cdots, \mathcal{N})$

adding spinor generators Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ to translations, rotations, boosts

$$\left\{ Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{J}}_{\dot{\alpha}} \right\} = 2\delta^{\mathrm{IJ}} \sigma^{\mu}_{\alpha\dot{\alpha}} P_{\mu}$$ broken in discrete space-time

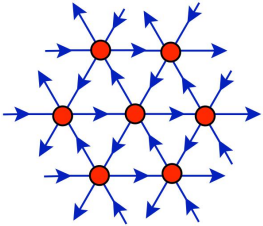
$\longrightarrow$ relevant susy-violating operators

Supersymmetry need not be *completely* broken on the lattice

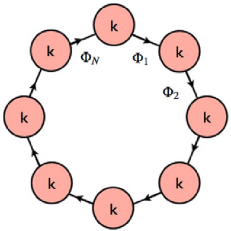
Preserve susy sub-algebra in discrete lattice space-time

$\Longrightarrow$ correct continuum limit with little or no fine tuning

Equivalent constructions from 'topological' twisting and dim'l deconstruction



Review:
Catterall–Kaplan–Ünsal,
arXiv:0903.4881



Need 2^d supersymmetries in d dimensions

$d = 4 \longrightarrow \mathcal{N} = 4$ super-Yang–Mills (SYM)

Twisting $\mathcal{N} = 4$ SYM

Intuitive 4d picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab}\gamma_a\gamma_b$$

with $a, b = 1, \cdots, 5$

Lorentz index $\times$ R-symmetry index $\Longrightarrow$ reps of 'twisted rotation group'

$$\mathrm{SO}(4)_{\mathrm{tw}} \equiv \mathrm{diag}\left[\mathrm{SO}(4)_{\mathrm{euc}} \otimes \mathrm{SO}(4)_R\right] \qquad \mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$$

Change of variables $\longrightarrow$ $\mathcal{Q}$s transform with integer 'spin' under $\mathrm{SO}(4)_{\mathrm{tw}}$

Twisting $\mathcal{N} = 4$ SYM

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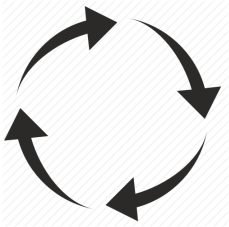
Discrete space-time

Can preserve closed sub-algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$



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with $a, b = 1, \cdots, 5$

Discrete space-time

Can preserve closed sub-algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$

Completing the twist

Fields also transform with integer spin under $\mathrm{SO}(4)_{\mathrm{tw}}$ — no spinors

$$\Psi \text{ and } \overline{\Psi} \longrightarrow \eta,\ \psi_a \text{ and } \chi_{ab}$$

$$A_\mu \text{ and } \Phi^{\mathrm{I}} \longrightarrow \text{complexified gauge field } \mathcal{A}_a \text{ and } \overline{\mathcal{A}}_a$$

$$\longrightarrow \mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1) \text{ gauge theory}$$

✓ $\mathcal{Q}$ interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\mathcal{Q}\,\mathcal{A}_a = \psi_a \qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad \mathcal{Q}\,\overline{\mathcal{A}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad \mathcal{Q}\,d = 0$$

bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Covariant derivatives $\longrightarrow$ finite difference operators

Complexified gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $\mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C})$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad \mathcal{Q}\,d = 0$$

Geometry: η on sites, ψ_a on links, etc.

Supersymmetric lattice action $(\mathcal{Q}S = 0)$ from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity

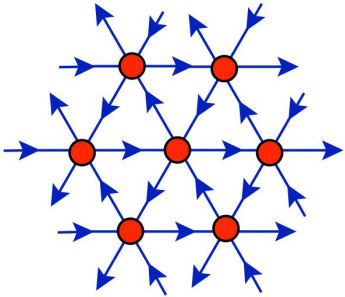
$$S = \frac{N}{4\lambda_{\text{lat}}} \text{Tr}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de}\right]$$

Five links in four dimensions $\longrightarrow$ A_4^* lattice

A_4^* $\sim$ 4d analog of 2d triangular lattice

Basis vectors linearly dependent and non-orthogonal

Large S_5 point group symmetry



S_5 irreps precisely match onto irreps of twisted $\mathrm{SO}(4)_{tw}$

$$\psi_a \longrightarrow \psi_\mu,\ \overline{\eta} \qquad \text{is} \qquad \mathbf{5} \longrightarrow \mathbf{4} \oplus \mathbf{1}$$

$$\chi_{ab} \longrightarrow \chi_{\mu\nu},\ \overline{\psi}_\mu \qquad \text{is} \qquad \mathbf{10} \longrightarrow \mathbf{6} \oplus \mathbf{4}$$

$S_5 \longrightarrow \mathrm{SO}(4)_{tw}$ in continuum limit restores $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Checkpoint

Analytic results for twisted $\mathcal{N} = 4$ SYM on A_4^* lattice

U(N) gauge invariance + $\mathcal{Q}$ + S_5 lattice symmetries

$\longrightarrow$ Moduli space preserved to all orders

$\longrightarrow$ One-loop lattice β function vanishes

$\longrightarrow$ Only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

[arXiv:1102.1725, arXiv:1306.3891, arXiv:1408.7067]

Not yet suitable for numerical calculations

Must regulate zero modes and flat directions, especially in U(1) sector

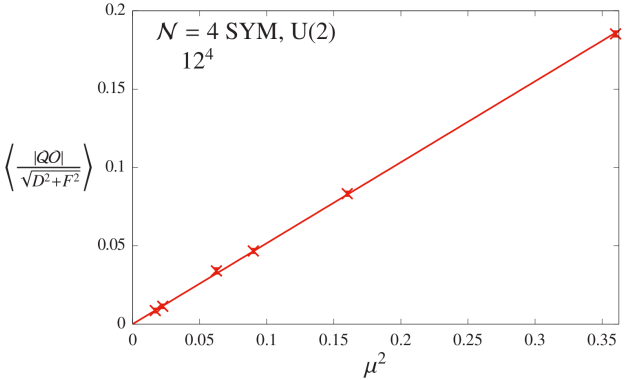
Two deformations stabilize lattice calculations

1) Add SU(N) scalar potential $\propto \mu^2 \sum_a \left(\mathrm{Tr}\left[\mathcal{U}_a \overline{\mathcal{U}}_a\right] - N\right)^2$

Softly breaks susy $\longrightarrow$ $\mathcal{Q}$-violating operators vanish $\propto \mu^2 \to 0$

Test via Ward identity violations

$$\mathcal{Q}\left[\eta \mathcal{U}_a \overline{\mathcal{U}}_a\right] \neq 0$$



Two deformations stabilize lattice calculations

2) Constrain U(1) plaquette determinant $\sim G \sum_{a<b} (\det \mathcal{P}_{ab} - 1)$

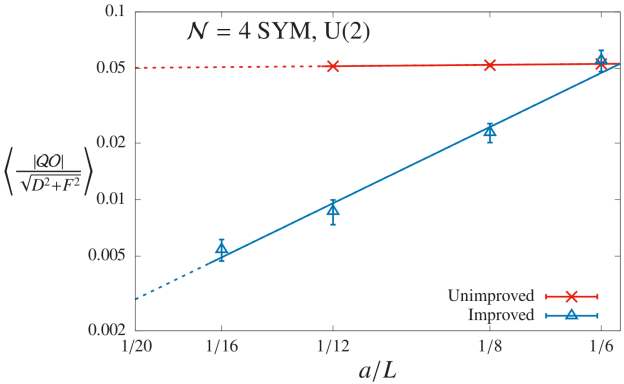
Implemented supersymmetrically by modifying auxiliary field equations of motion

Test via Ward identity violations

$$\mathcal{Q}\left[\eta \mathcal{U}_a \overline{\mathcal{U}}_a\right] \neq 0$$

Log–log axes

$\longrightarrow$ violations $\propto (a/L)^2$



Public code for supersymmetric lattice field theories

so that the full improved action becomes

$$S_{\rm imp} = S'_{\rm exact} + S_{\rm closed} + S'_{\rm soft} \tag{18}$$

$$S'_{\rm exact} = \frac{N}{4\lambda_{\rm lat}} \sum_n {\rm Tr}\Bigg[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_a\psi_a(n) + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a(n) + G\sum_{a\neq b}(\det\mathcal{P}_{ab}(n) - 1)\,\mathbb{I}_N\right)^2\Bigg] - S_{\rm det}$$

$$S_{\rm det} = \frac{N}{4\lambda_{\rm lat}} G \sum_n {\rm Tr}\left[\eta(n)\right] \sum_{a\neq b}\left[\det\mathcal{P}_{ab}(n)\right] {\rm Tr}\left[\mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n+\widehat{\mu}_b)\psi_a(n+\widehat{\mu}_b)\right]$$

$$S_{\rm closed} = -\frac{N}{16\lambda_{\rm lat}} \sum_n {\rm Tr}\left[\epsilon_{abcde}\,\chi_{de}(n+\widehat{\mu}_a+\widehat{\mu}_b+\widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_c\chi_{ab}(n)\right],$$

$$S'_{\rm soft} = \frac{N}{4\lambda_{\rm lat}}\mu^2 \sum_n\sum_a\left(\frac{1}{N}{\rm Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right)^2$$

$\gtrsim$100 inter-node data transfers in the fermion operator — non-trivial...

Public parallel code to reduce barriers to entry: github.com/daschaich/susy

Evolved from MILC QCD code, user guide in arXiv:1410.6971

2d and 3d maximal SYM

arXiv:1709.07025 and arXiv:2010.00026

Naive dimensional reduction $\longrightarrow$ skewed tori

$r_L \times r_\beta$ with $r_\beta = \sqrt{\lambda}/T$ and four scalar $\mathcal{Q}$

$r_1 \times r_2 \times r_\beta$ with $r_\beta = \lambda/T$ and two scalar $\mathcal{Q}$

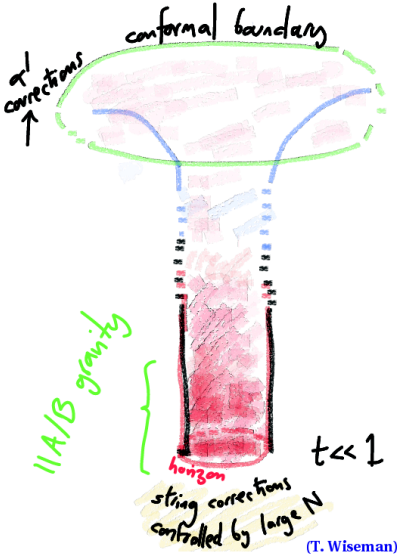
Thermal boundary conditions

$\longrightarrow$ dimensionless temperature $t = 1/r_\beta$

Low temperatures t at large N

$\updownarrow$

Black branes in dual supergravity



(T. Wiseman)

2d $\mathcal{N} = (8,8)$ SYM phase diagram

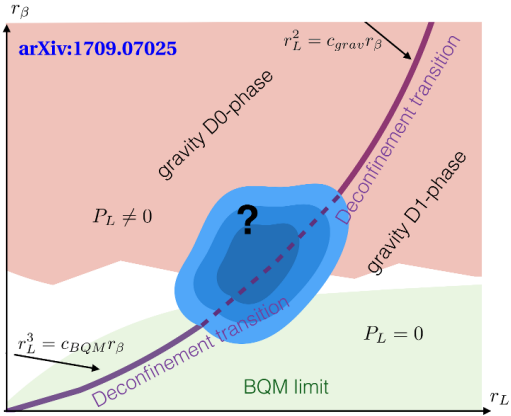
First-order transitions predicted from bosonic QM at high t $(r_\beta \ll 1)$
from holography at low t $(r_\beta \gg 1)$

For decreasing r_L at large N

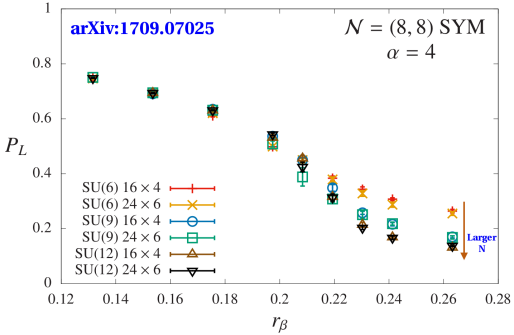
homogeneous black string (D1)
$\longrightarrow$ localized black hole (D0)

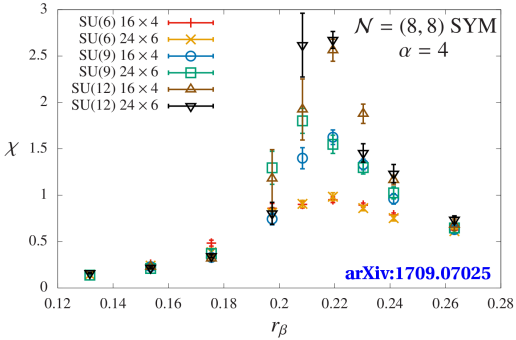
$\updownarrow$

"spatial deconfinement"
signalled by Wilson line P_L



Spatial deconfinement transition signals — high-t example





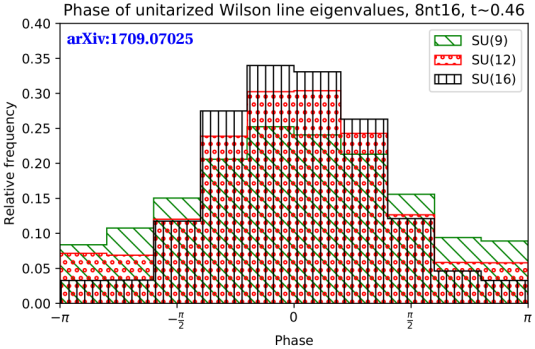
Fix aspect ratio $\alpha = r_L/r_\beta = 4$ Check 16×4 vs. 24×6 lattices agree

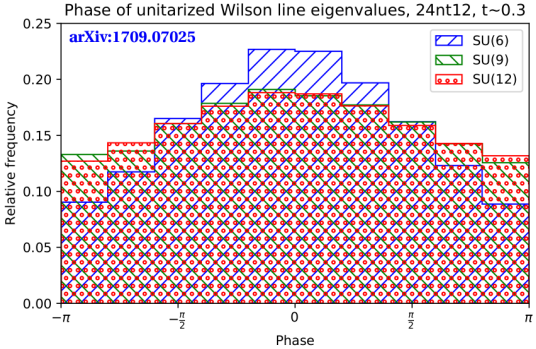
Peaks in $\mathrm{Tr}P_L$ susceptibility match change in its magnitude,

grow with size of SU(N) gauge group, comparing $N = 6, 9, 12$

Wilson line eigenvalues for low t

Large-N eigenvalue phase distribution also signals spatial deconfinement





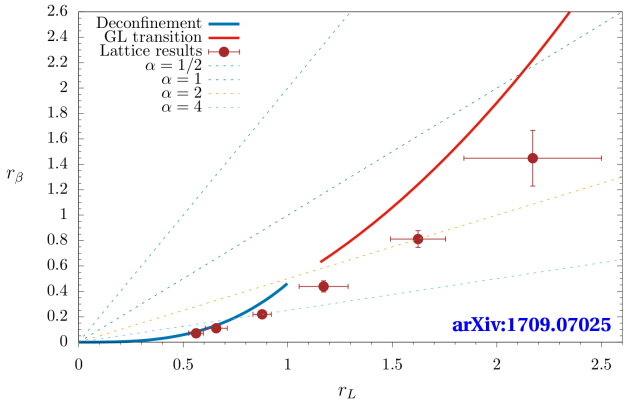
Left: $\alpha = 1/2$ distributions more localized as N increases $\longrightarrow$ D0 black hole

Right: $\alpha = 2$ distributions more uniform as N increases $\longrightarrow$ D1 black string

Lattice results for 2d $\mathcal{N} = (8,8)$ SYM phase diagram

Good agreement with bosonic QM at high temperatures ($\alpha \gtrsim 4$)

Harder to control low-temperature uncertainties (larger $N > 16$ should help)



Overall consistent with holography

Comparing multiple lattice sizes and $6 \leq N \leq 16$

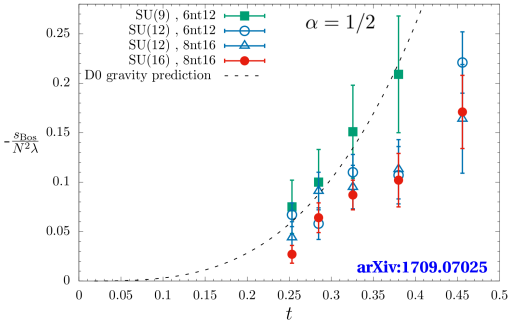
Controlled extrapolations not yet attempted in 2d

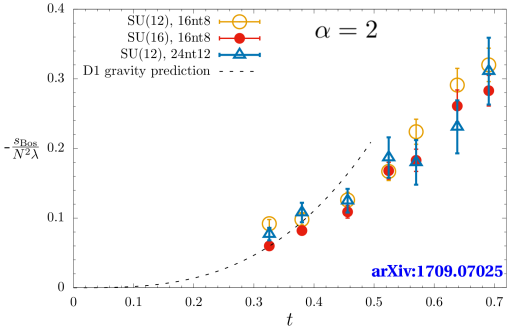
Holographic black hole energies

Lattice results consistent with leading expectation for sufficiently low $t \lesssim 0.4$

Similar behavior $\longrightarrow$ difficult to distinguish phases

$\propto t^{3.2}$ for small-r_L D0 phase $\qquad$ $\propto t^{3}$ for large-r_L D1 phase



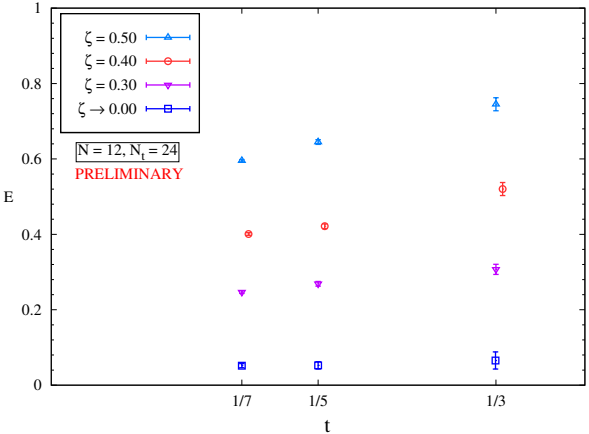


Compare with $\mathcal{N} = (2,2)$ SYM

arXiv:2109.01001

Much simpler twisted formulation: $Q = 4$ supercharges $\{\mathcal{Q}, \mathcal{Q}_a, \mathcal{Q}_{ab}\}$

$\longrightarrow$ site / link / plaquette fermions $\{\eta, \psi_a, \chi_{ab}\}$ on square lattice $(a, b = 1, 2)$



Work by Navdeep Singh Dhindsa

Prelim. $\mu^2 \to 0$ extrapolations

for $r_L = r_\beta \longleftrightarrow \alpha = 1$

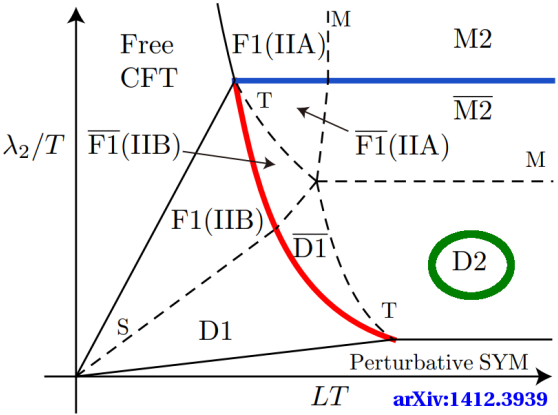
Energy independent of $t \lesssim 0.33$

vs. $\sim t^3$ for $\mathcal{N} = (8,8)$ SYM

3d maximal SYM

Holography $\longrightarrow$ much richer low-t phase diagram than for 2d $\mathcal{N} = (8,8)$ SYM

For now consider simplest homogeneous black D2-branes $\longrightarrow$ $r_1 = r_2 = r_\beta$

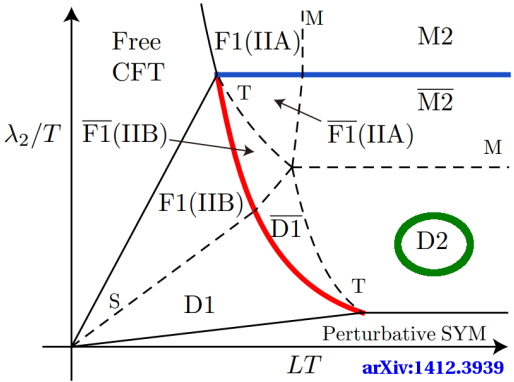


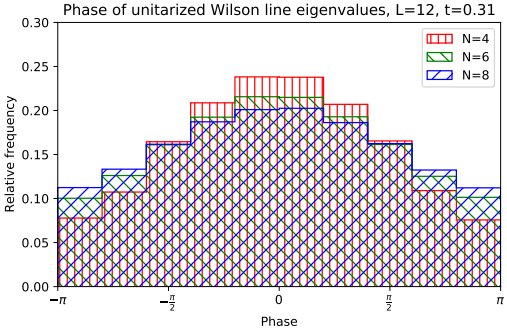
arXiv:1412.3939

Homogeneous D2 phase

Lattice volume $(L/a)^3 \longrightarrow$ continuum limit $L/a \to \infty$ with fixed $t = 1/r_\beta = L/\lambda$

Homogeneous D2-branes $\longleftrightarrow$ uniform Wilson line eigenvalue phases at large N



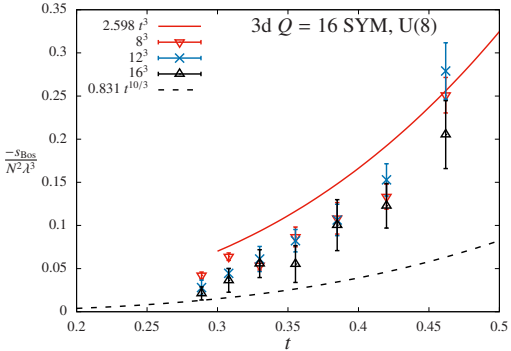


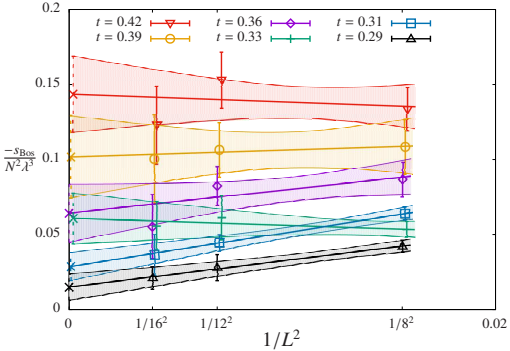
Holographic black brane energies and continuum extrapolation

Lattice volume $(L/a)^3$ with fixed $N = 8$

$\longrightarrow$ results approach leading holographic expectation $\propto t^{10/3}$ for low $t \lesssim 0.4$

Carry out first $L/a \to \infty$ continuum extrapolations

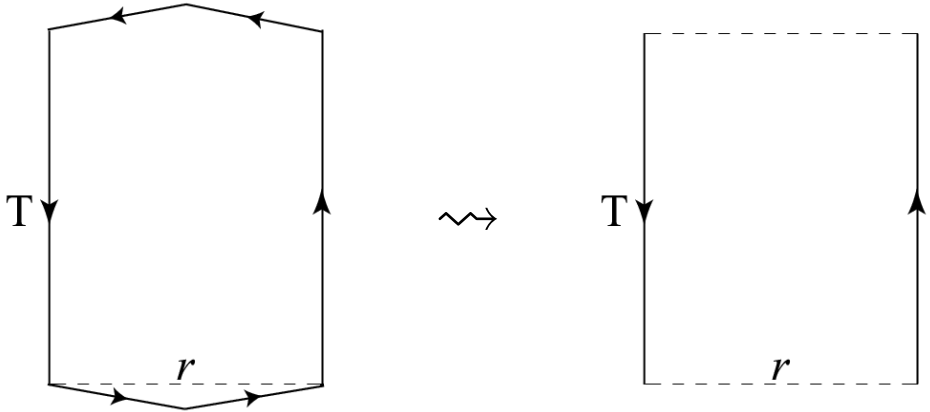




4d $\mathcal{N} = 4$ SYM static potential $V(r)$

Static probes $\longrightarrow$ $r \times T$ Wilson loops $W(r, T) \propto e^{-V(r)\,T}$

Coulomb gauge trick reduces A_4^* lattice complications



Static potential is Coulombic at all λ

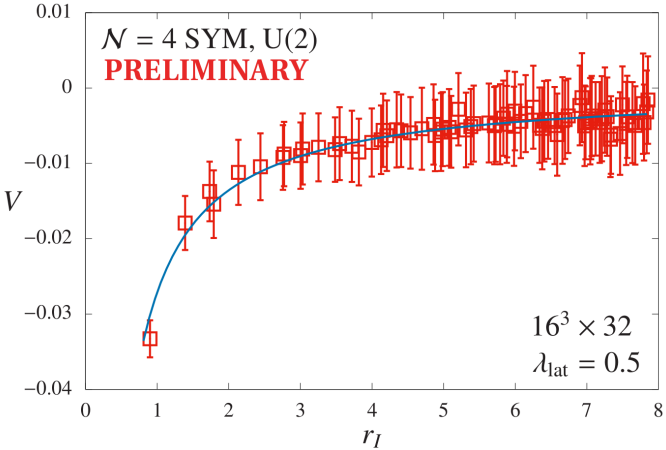
Fits to confining $V(r) = A - C/r + \sigma r \longrightarrow$ vanishing string tension σ

Therefore fit

$$V(r) = A - C/r$$

to find Coulomb coefficient $C(\lambda)$

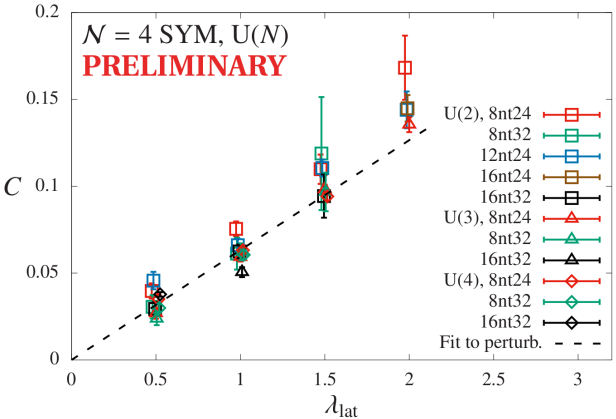
Discretization artifacts reduced by tree-level improved analysis



Coupling dependence of Coulomb coefficient

Continuum perturbation theory $\longrightarrow$ $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

Holography $\longrightarrow$ $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$ and $\lambda \to \infty$ with $\lambda \ll N$



Again comparing different volumes and $N = 2, 3, 4$

For $\lambda_{\text{lat}} \leq 2$, consistent with leading-order perturbation theory

Konishi operator scaling dimension Δ_K

$\mathcal{O}_K(x) = \sum_{\mathrm{I}} \mathrm{Tr}\,[\Phi^{\mathrm{I}}(x)\Phi^{\mathrm{I}}(x)]$ is simplest conformal primary operator

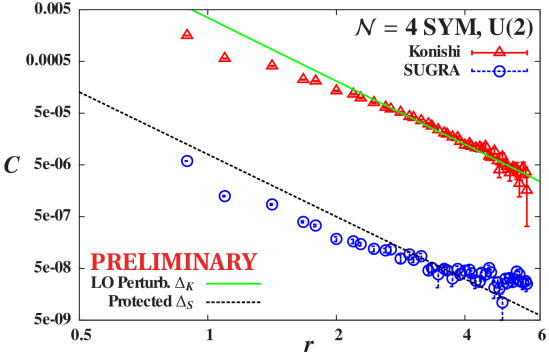
Scaling dimension $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$ investigated through perturbation theory (& S duality), holography, conformal bootstrap

$$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

'SUGRA' is 20′ op., $\Delta_S = 2$

Work in progress to compare:

- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG



Konishi operator scaling dimension Δ_K

Lattice scalars $\varphi(n)$ from polar decomposition $\mathcal{U}_a(n) = e^{\varphi_a(n)} U_a(n)$

$$\mathcal{O}_K^{\text{lat}}(n) = \sum_a \text{Tr}\left[\varphi_a(n)\varphi_a(n)\right] - \text{vev} \qquad \mathcal{O}_S^{\text{lat}}(n) \sim \text{Tr}\left[\varphi_a(n)\varphi_b(n)\right]$$

$$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

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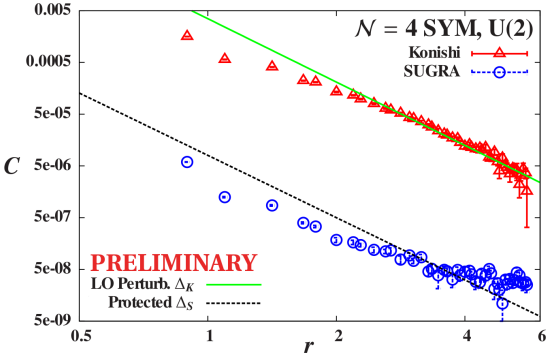
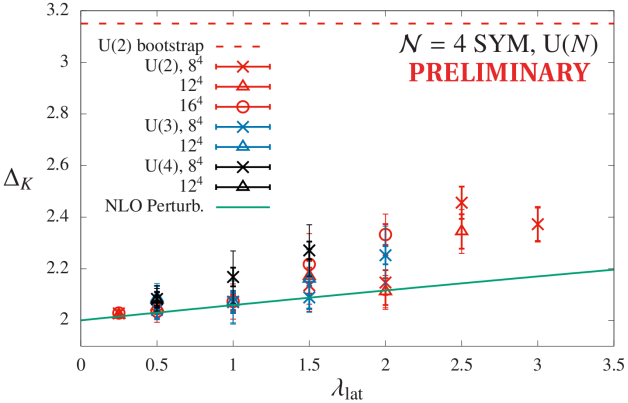


Preliminary Δ_K results from Monte Carlo RG

Analyzing both $\mathcal{O}_K^{\text{lat}}$ and $\mathcal{O}_S^{\text{lat}}$

Imposing protected $\Delta_S = 2$
$\longrightarrow \Delta_K(\lambda)$ looks perturbative

Systematic uncertainties from
different amounts of smearing



Complication from twisting $\text{SO}(4)_R \subset \text{SO}(6)_R$

$\mathcal{O}_K^{\text{lat}}$ mixes with $\text{SO}(4)_R$-singlet part of $\text{SO}(6)_R$-nonsinglet $\mathcal{O}_S$
$\longrightarrow$ disentangle via variational analyses

Supplement: Pushing $\mathcal{N} = 4$ SYM to stronger coupling

✓ Reproduce reliable 4d results in perturbative regime

$\longrightarrow$ Check holographic predictions and access new domains

Sign problem seems to become obstruction

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}] \;\; \mathcal{O} \;\; e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]} \;\; \mathrm{pf}\, \mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

Complex pfaffian $\mathrm{pf}\,\mathcal{D} = |\mathrm{pf}\,\mathcal{D}| e^{i\alpha}$ complicates importance sampling

We phase quench, $\mathrm{pf}\,\mathcal{D} \longrightarrow |\mathrm{pf}\,\mathcal{D}|$, need to reweight $\langle \mathcal{O} \rangle = \dfrac{\langle \mathcal{O} e^{i\alpha} \rangle_{\mathrm{pq}}}{\langle e^{i\alpha} \rangle_{\mathrm{pq}}}$

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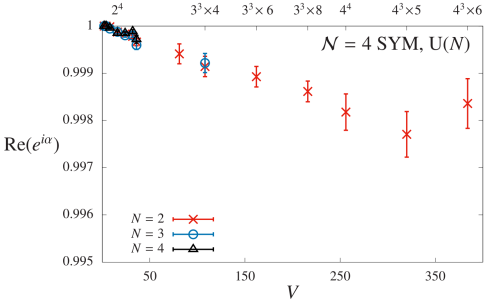
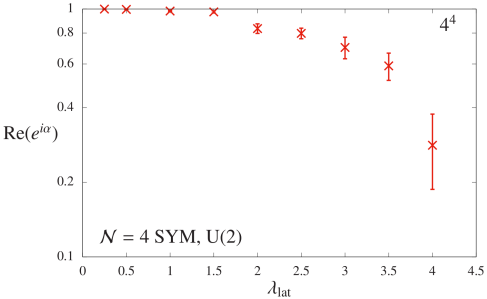
$$\Longrightarrow \left\langle e^{i\alpha} \right\rangle_{\mathrm{pq}} = \frac{\mathcal{Z}}{\mathcal{Z}_{\mathrm{pq}}}$$ quantifies severity of sign problem

$\mathcal{N} = 4$ SYM sign problem

Fix $\lambda_{\text{lat}} = g^2_{\text{lat}} N = 0.5$

Pfaffian nearly real positive
for all accessible volumes





Fix 4^4 volume

Fluctuations increase with coupling

Signal-to-noise
becomes obstruction for $\lambda_{\text{lat}} \gtrsim 4$

Supplement: Supersymmetric QCD

Add matter multiplets $\longrightarrow$ investigate electric–magnetic dualities,
dynamical supersymmetry breaking and more

Quiver construction based on twisted SYM [arXiv:1505.00467]
preserves susy sub-algebra in $(d-1)$ dims. to reduce fine-tuning

Quiver superQCD from twisted SYM

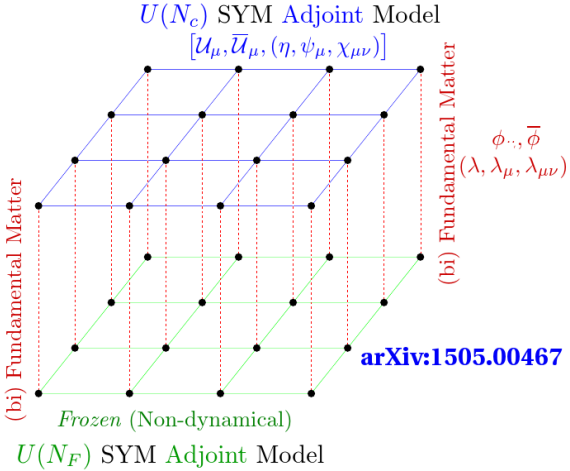
2-slice lattice SYM
with U(N) $\times$ U(F) gauge group

Adj. fields on each slice

Bi-fundamental in between

Decouple U(F) slice

$\longrightarrow$ U(N) SQCD in ($d-1$) dims.
with F fund. hypermultiplets



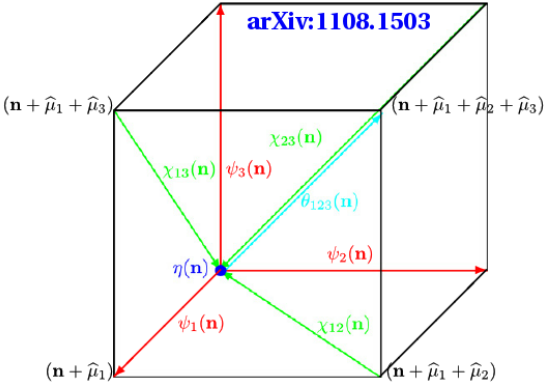
First check 3d SYM $\longrightarrow$ 2d superQCD then new 4d SYM $\longrightarrow$ 3d superQCD

First step: 8-supercharge SYM in 3d

Simpler twisted formulation

$Q = 8$ supercharges $\{\mathcal{Q}, \mathcal{Q}_a, \mathcal{Q}_{ab}, \mathcal{Q}_{abc}\}$ with $a, b = 1, \cdots, 3$

$\longrightarrow$ site / link / plaquette / cube fermions $\{\eta, \psi_a, \chi_{ab}, \theta_{abc}\}$ on simple cubic lattice



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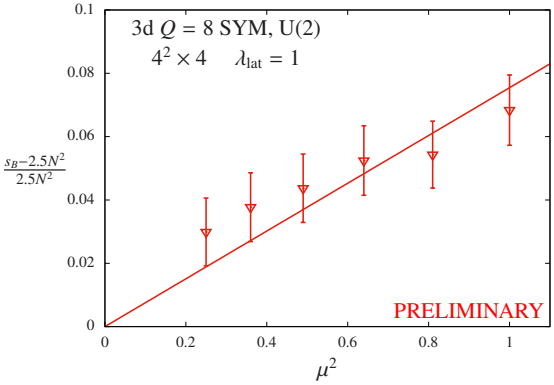
Work by Angel Sherletov

Parallel code developed

Initial tests passed

$\longrightarrow$ larger-scale calculations



Recap: An exciting time for lattice supersymmetry

✓ Preserve (some) susy in discrete space-time

$\longrightarrow$ practical lattice $\mathcal{N} = 4$ SYM, public code available

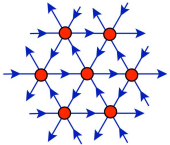
Reproduce reliable analytic results

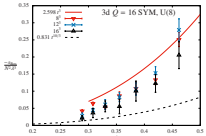
✓ 2d and 3d thermodynamics consistent with holography

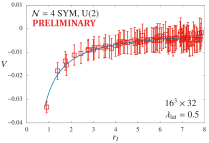
✓ Perturbative $\mathcal{N} = 4$ SYM static potential Coulomb coefficient $C(\lambda)$

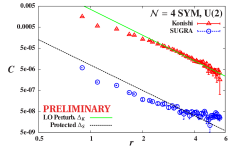
and Konishi operator scaling dimension $\Delta_K(\lambda)$

Access new domains $\longrightarrow$ sign problems, supersymmetric QCD and more...









Thanks for your attention! Any further questions?

Collaborators

Raghav Jha, Anosh Joseph, Angel Sherletov, Toby Wiseman

also Georg Bergner, Simon Catterall, Poul Damgaard, Joel Giedt

Funding and computing resources

UK Research and Innovation

LIVERPOOL THEORETICAL PHYSICS

DEPARTMENT OF ENERGY · UNITED STATES OF AMERICA

USQCD

Backup: Breakdown of Leibniz rule on the lattice

$\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$ is problematic

$\Longrightarrow$ try finite difference $\partial\phi(x) \longrightarrow \Delta\phi(x) = \frac{1}{a}\left[\phi(x+a) - \phi(x)\right]$

Crucial difference between ∂ and Δ

$$\Delta\left[\phi\eta\right] = a^{-1}\left[\phi(x+a)\eta(x+a) - \phi(x)\eta(x)\right]$$
$$= \left[\Delta\phi\right]\eta + \phi\Delta\eta + a\left[\Delta\phi\right]\Delta\eta$$

Full supersymmetry requires Leibniz rule $\partial\left[\phi\eta\right] = \left[\partial\phi\right]\eta + \phi\partial\eta$

only recoverd in $a \to 0$ continuum limit for any local finite difference

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Supersymmetry vs. locality 'no-go' theorems

by Kato–Sakamoto–So [arXiv:0803.3121] and Bergner [arXiv:0909.4791]

Complicated constructions to balance locality vs. supersymmetry

Non-ultralocal product operator $\longrightarrow$ lattice Leibniz rule but not gauge invariance

D'Adda–Kawamoto–Saito, arXiv:1706.02615

Cyclic Leibniz rule $\longrightarrow$ partial lattice supersymmetry but only (0+1)d QM so far

Kadoh–Kamei–So, arXiv:1904.09275

Backup: $\mathcal{N} = 4$ SYM in a nutshell

Arguably simplest non-trivial 4d QFT $\longrightarrow$ dualities, amplitudes, ...

SU(N) gauge theory with $\mathcal{N} = 4$ fermions Ψ^{I} and 6 scalars Φ^{IJ},
all massless and in adjoint rep.

Symmetries relate coefficients of kinetic, Yukawa and Φ^4 terms

Maximal 16 supersymmetries Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ $\quad \mathrm{I} = 1, \cdots, 4$
transform under global SU(4) $\sim$ SO(6) R symmetry

Conformal $\longrightarrow$ β function is zero for all values of $\lambda = g^2 N$

Backup: Complexified gauge field from twisting

Combining A_μ and $\Phi^{\mathrm{I}} \longrightarrow \mathcal{A}_a$ and $\overline{\mathcal{A}}_a$

produces $\mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1)$ gauge theory

Complicates lattice action but needed so that $\mathcal{Q}\,\mathcal{A}_a = \psi_a$

Further motivation: Under $\mathrm{SO}(d)_{\mathrm{tw}} = \mathrm{diag}\left[\mathrm{SO}(d)_{\mathrm{euc}} \otimes \mathrm{SO}(d)_R\right]$

$$A_\mu \sim \text{vector} \otimes \text{scalar} = \text{vector}$$
$$\Phi^{\mathrm{I}} \sim \text{scalar} \otimes \text{vector} = \text{vector}$$

Easiest to see in 5d, then dimensionally reduce

$$\mathcal{A}_a = A_a + i\Phi_a \longrightarrow (A_\mu, \phi) + i(\Phi_\mu, \overline{\phi})$$

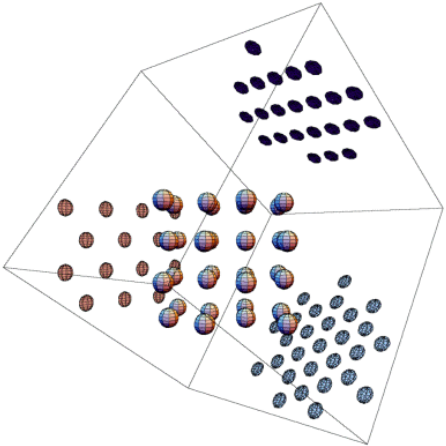
Backup: A_4^* lattice from five dimensions

Again dimensionally reduce, treating all five gauge links symmetrically

Start with hypercubic lattice
in 5d momentum space

Symmetric constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in position space



Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

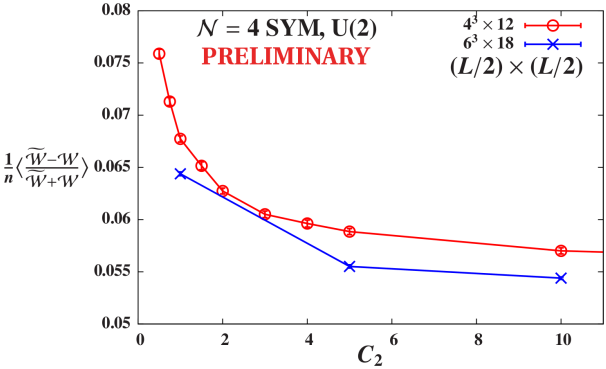
"$\mathcal{Q}$ + discrete $R_a \subset \mathrm{SO}(4)_{\mathrm{tw}}$ = $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$"

[arXiv:1306.3891]

Test R_a on Wilson loops

$$\widetilde{\mathcal{W}}_{ab} \equiv R_a \mathcal{W}_{ab}$$

Tune coeff. c_2 of d^2 term in action for fastest restoration towards continuum limit

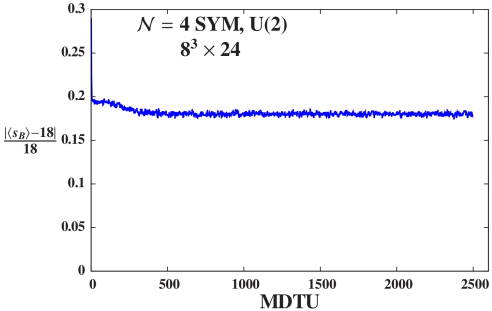


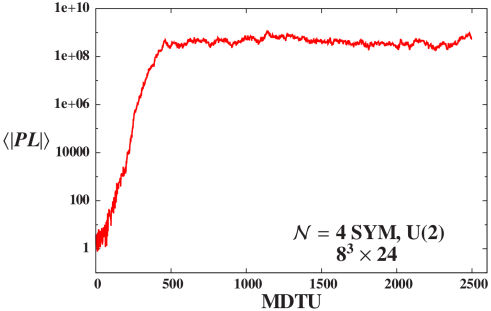
Backup: Problem with SU(N) flat directions

$\mu^2/\lambda_{\text{lat}}$ too small $\longrightarrow$ $\mathcal{U}_a$ can move far from continuum form $\mathbb{I}_N + \mathcal{A}_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 2.5$ on $8^3 \times 24$ volume

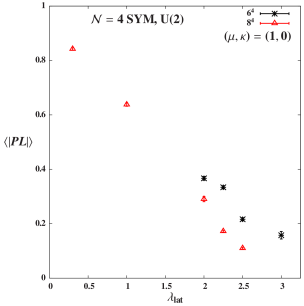
Left: Bosonic action stable $\sim$18% off its supersymmetric value
Right: (Complexified) Polyakov loop wanders off to $\sim 10^9$

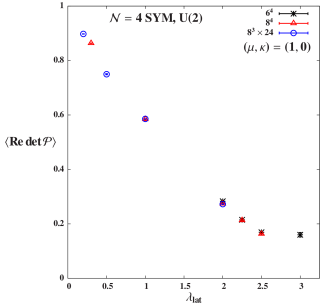


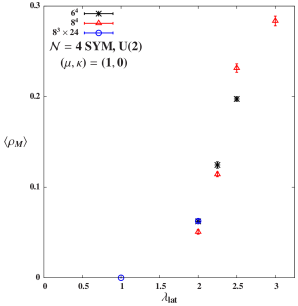


Backup: Problem with U(1) flat directions

Monopole condensation $\longrightarrow$ confined lattice phase not present in continuum







Around the same $2\lambda_{\rm lat} \approx 2\ldots$

Left: Polyakov loop falls towards zero

Center: Plaquette determinant falls towards zero

Right: Density of U(1) monopole world lines becomes non-zero

Backup: Naively regulating U(1) flat directions

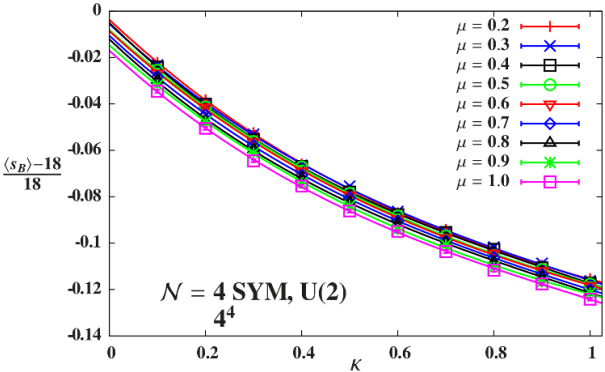
In earlier work we added another soft $\mathcal{Q}$-breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}}\mu^2 \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2 + \kappa \sum_{a<b} \left|\det \mathcal{P}_{ab} - 1\right|^2$$

More sensitivity to κ than to μ^2

Showing $\mathcal{Q}$ Ward identity from bosonic action

$$\langle s_B \rangle = 9N^2/2$$

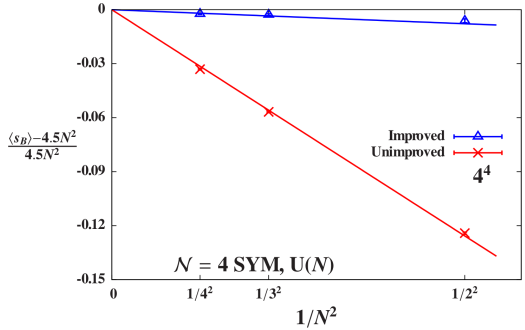


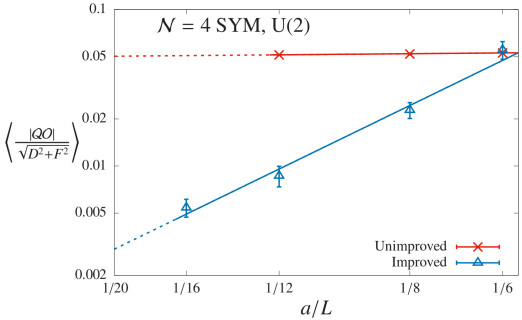
Backup: Better regulating U(1) flat directions

$$S = \frac{N}{4\lambda_{\text{lat}}}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\left\{\overline{\mathcal{D}}_a\mathcal{U}_a + G\sum_{a<b}[\det\mathcal{P}_{ab} - 1]\,\mathbb{I}_N\right\} - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V\right]$$

$\mathcal{Q}$ Ward identity violations scale $\propto 1/N^2$ (**left**) and $\propto (a/L)^2$ (**right**)

$\sim$ effective '$\mathcal{O}(a)$ improvement' since $\mathcal{Q}$ forbids all dim-5 operators





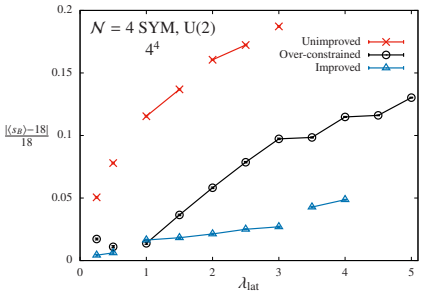
Backup: Supersymmetric moduli space modification [arXiv:1505.03135]

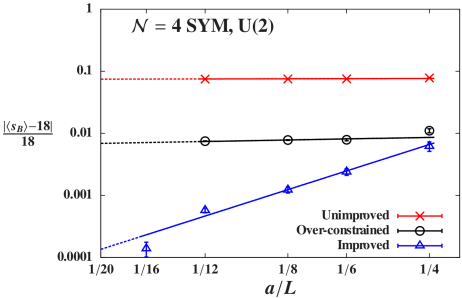
Method to impose $\mathcal{Q}$-invariant constraints on generic site operator $\mathcal{O}(n)$

Modify auxiliary field equations of motion $\longrightarrow$ moduli space

$$d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) \qquad \longrightarrow \qquad d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

Including both U(1) and SU(N) $\in \mathcal{O}(n)$ over-constrains system





Backup: Dimensional reduction to 2d $\mathcal{N} = (8,8)$ SYM

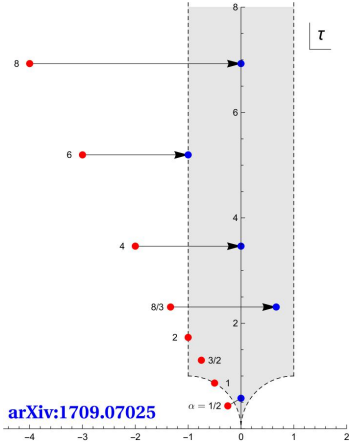
Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

$A_4^* \longrightarrow A_2^*$ (triangular) lattice

Torus **skewed** depending on $\alpha = L/N_t$

Modular transformation into fundamental domain
$\longrightarrow$ some skewed tori actually rectangular

Also need to stabilize compactified links
to ensure broken center symmetries

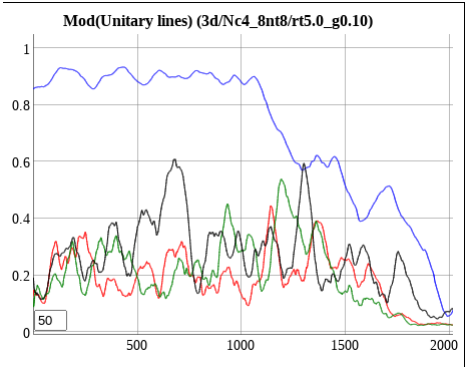


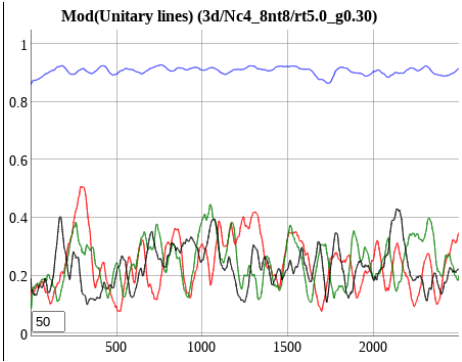
arXiv:1709.07025

Backup: Stabilizing compactified links

Add potential $\propto \mathrm{Tr}\left[(\varphi - \mathbb{I}_N)^\dagger (\varphi - \mathbb{I}_N)\right]$ to break center symmetry in reduced dir(s)

($\sim$Kaluza–Klein rather than Eguchi–Kawai reduction)

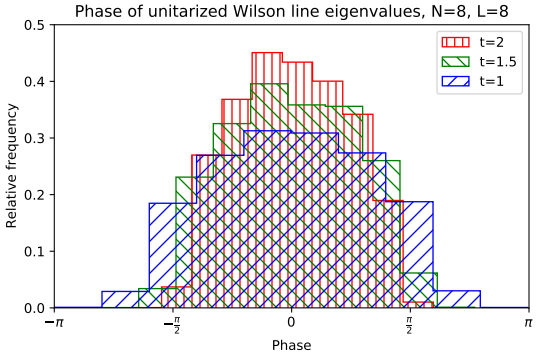


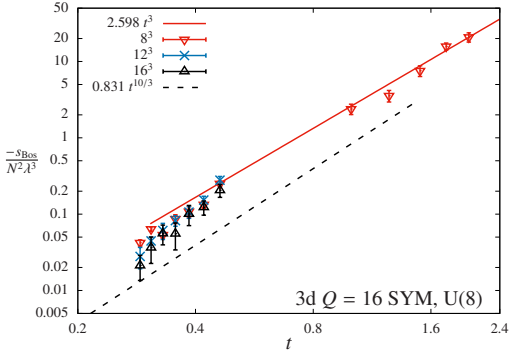


Backup: High-temperature ($t \gtrsim 1$) 3d maximal SYM

Wilson line eigenvalue phases localized rather than uniform (**left**)

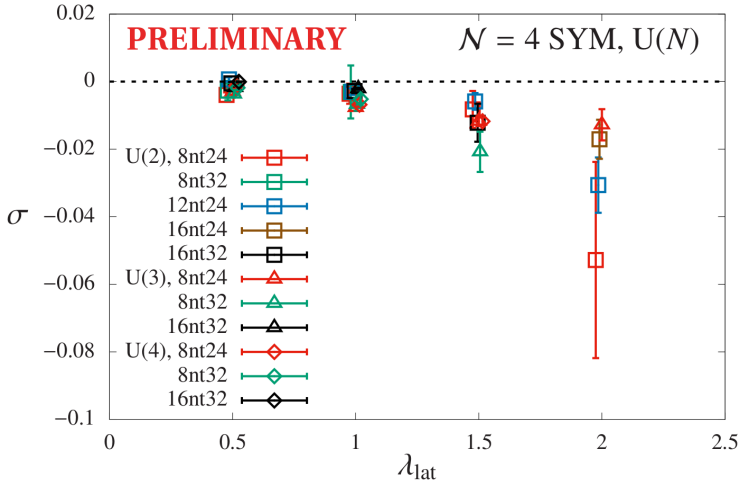
Thermodynamics consistent with weak-coupling expectation $\propto t^3$ (**right**)





Backup: Static potential is Coulombic at all λ

String tension σ from fits to confining form $V(r) = A - C/r + \sigma r$

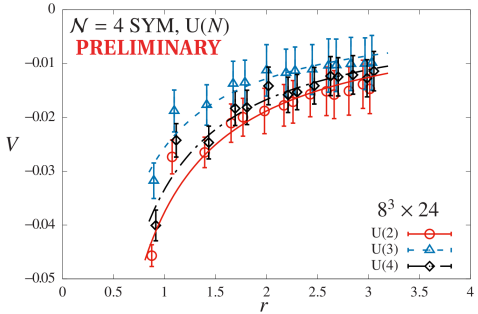


Slightly negative values flatten $V(r_l)$ for $r_l \lesssim L/2$

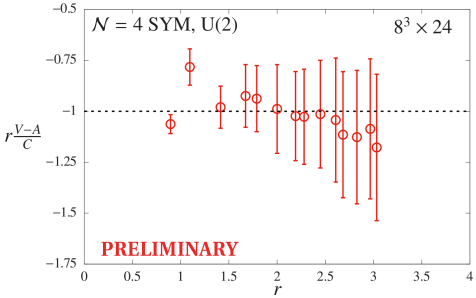
$\sigma \to 0$ as accessible range of r_l increases on larger volumes

Backup: Discretization artifacts in static potential

Discretization artifacts visible at short distances
where Coulomb term in $V(r) = A - C/r$ is most significant



$\rightsquigarrow$



Danger of distorting Coulomb coefficient C

Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential

Associate $V(r_\nu)$ data with 'r_l' from Fourier transform of gluon propagator

Recall $\dfrac{1}{4\pi^2 r^2} = \displaystyle\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{e^{ir_\nu k_\nu}}{k^2}$ where $\dfrac{1}{k^2} = G(k_\nu)$ in continuum

$$A_4^* \text{ lattice} \longrightarrow \frac{1}{r_l^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4\widehat{k}}{(2\pi)^4} \frac{\cos\left(ir_\nu \widehat{k}_\nu\right)}{4\sum_{\mu=1}^{4} \sin^2\left(\widehat{k}\cdot\widehat{e}_\mu \,/\, 2\right)}$$

Tree-level lattice propagator from arXiv:1102.1725

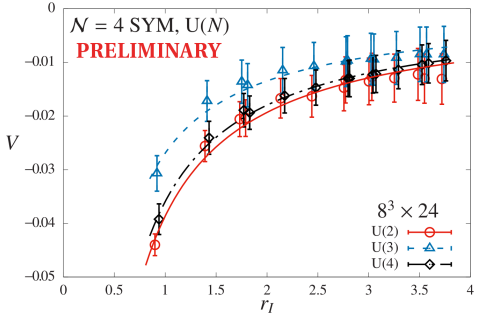
$\widehat{e}_\mu$ are A_4^* lattice basis vectors;

momenta $\widehat{k} = \frac{2\pi}{L}\sum_{\mu=1}^{4} n_\mu \widehat{g}_\mu$ depend on dual basis vectors

Backup: Tree-level-improved static potential

$$\frac{1}{r_I^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4\widehat{k}}{(2\pi)^4} \frac{\cos\left(i r_\nu \widehat{k}_\nu\right)}{4\sum_{\mu=1}^{4} \sin^2\left(\widehat{k}\cdot\widehat{e}_\mu\,/\,2\right)}$$

$\longrightarrow$ significantly reduced discretization artifacts



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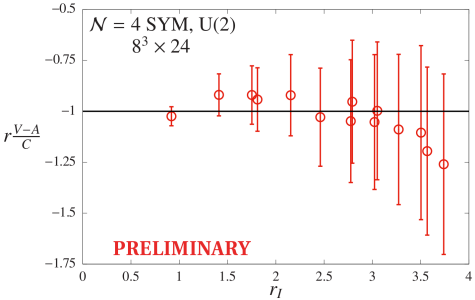


Backup: Scaling dimensions from MCRG stability matrix

Lattice system: $H = \sum_i c_i \, \mathcal{O}_i$ (infinite sum)

Couplings flow under RG blocking $\longrightarrow$ $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Conformal fixed point $\longrightarrow$ $H^\star = R_b H^\star$ with couplings $c_i^\star$

Linear expansion around fixed point $\longrightarrow$ stability matrix $T_{ik}^\star$

$$c_i^{(n)} - c_i^\star = \sum_k \left. \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \right|_{H^\star} \left(c_k^{(n-1)} - c_k^\star \right) \equiv \sum_k T_{ik}^\star \left(c_k^{(n-1)} - c_k^\star \right)$$

Correlators of $\mathcal{O}_i$, $\mathcal{O}_k$ $\longrightarrow$ elements of stability matrix [Swendsen, 1979]

Eigenvalues of $T_{ik}^\star$ $\longrightarrow$ scaling dimensions of corresponding operators

Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $\mathcal{Q}$ and S_5 symmetries $\longleftrightarrow$ geometric structure

Simple transformation constructed in arXiv:1408.7067

$$\mathcal{U}'_a(n') = \xi\, \mathcal{U}_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) \qquad \eta'(n') = \eta(n)$$

$$\psi'_a(n') = \xi\, [\psi_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) + \mathcal{U}_a(n)\psi_a(n+\widehat{\mu}_a)] \qquad \text{etc.}$$

Doubles lattice spacing $a \longrightarrow a' = 2a$, with tunable rescaling factor ξ

Scalar fields from polar decomposition $\mathcal{U}(n) = e^{\varphi(n)} U(n)$

$\Longrightarrow$ shift $\varphi \longrightarrow \varphi + \log \xi$ to keep blocked U unitary

$\mathcal{Q}$-preserving RG transformation needed

to show only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Backup: Smearing for Konishi analyses

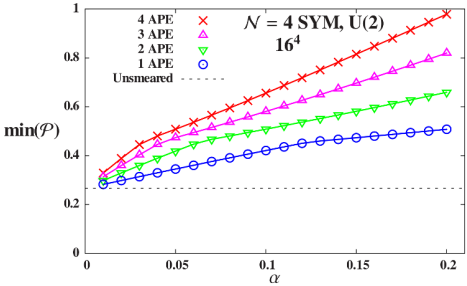
Smear to enlarge (MCRG or variational) operator basis

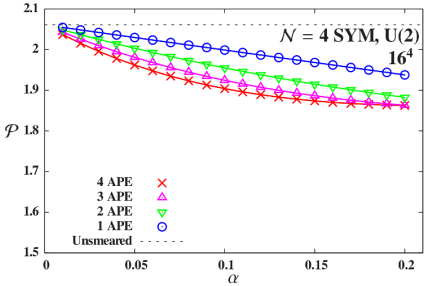
APE-like smearing: $\quad — \quad \longrightarrow \quad (1-\alpha)— \quad + \quad \frac{\alpha}{8}\sum \sqcap,$

staples built from unitary parts of links but no final unitarization

Average plaquette stable upon smearing (**right**),

minimum plaquette steadily increases (**left**)



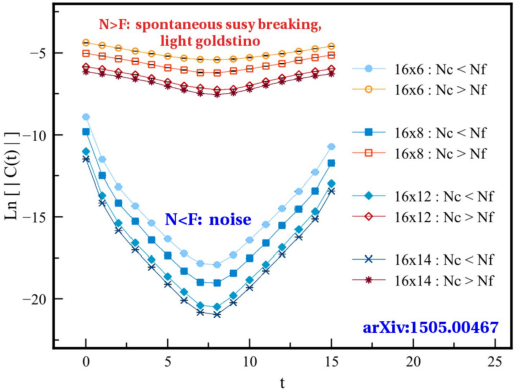


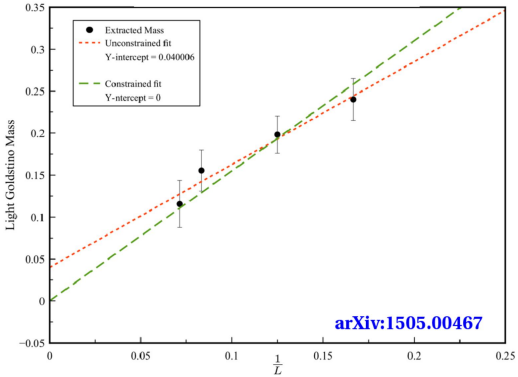
Backup: Dynamical susy breaking in 2d lattice superQCD

U(N) superQCD with F fundamental hypermultiplets

Observe spontaneous susy breaking only for $N > F$, as expected

Catterall–Veernala, arXiv:1505.00467





Backup: More on dynamical susy breaking

Spontaneous susy breaking means $\langle 0\,|H|\,0\rangle > 0$ or equivalently $\langle \mathcal{Q}\mathcal{O}\rangle \neq 0$

Twisted superQCD auxiliary field e.o.m. $\longleftrightarrow$ Fayet–Iliopoulos D-term potential

$$d = \overline{\mathcal{D}}_a \mathcal{U}_a + \sum_{i=1}^{F} \phi_i \overline{\phi}_i - r\mathbb{I}_N \qquad \longleftrightarrow \qquad \mathrm{Tr}\left[\left(\sum\nolimits_i \phi_i \overline{\phi}_i - r\mathbb{I}_N\right)^2\right] \in H$$

Have $F \times N$ scalar vevs to zero out $N \times N$ matrix

$\longrightarrow N > F$ suggests susy breaking, $\langle 0\,|H|\,0\rangle > 0 \longleftrightarrow \langle \mathcal{Q}\eta\rangle = \langle d\rangle \neq 0$