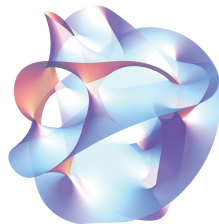
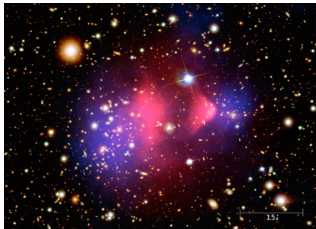
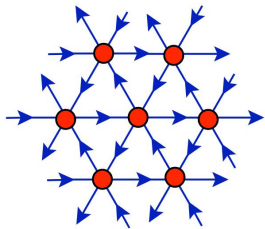


# Broader applications of lattice field theory

David Schaich (University of Liverpool)



UKLFT kick-off event, 24 March 2021

# Overview and plan

Lattice field theory is a broadly applicable tool  
to study strongly coupled quantum field theories

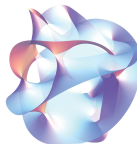
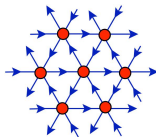
Composite dark matter and gravitational waves [2006.16429]

Composite Higgs and near-conformal dynamics [2007.01810]

Supersymmetry and holographic duality [2010.00026]

These slides: [davidschaich.net/talks/2103UKLFT.pdf](https://davidschaich.net/talks/2103UKLFT.pdf)

Interaction encouraged — complete coverage unnecessary

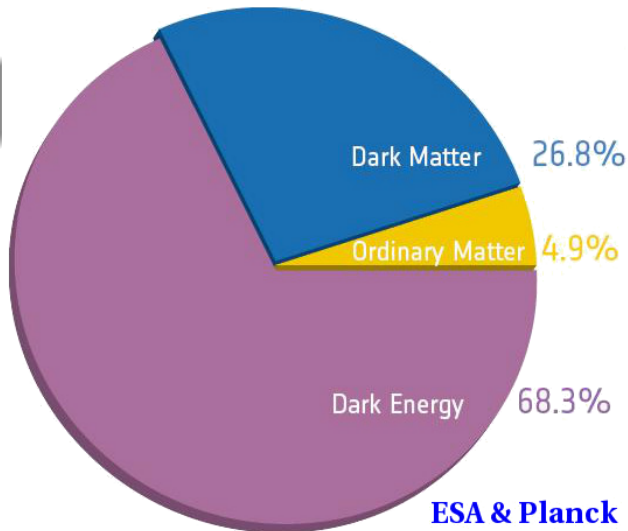


# Application: Dark matter

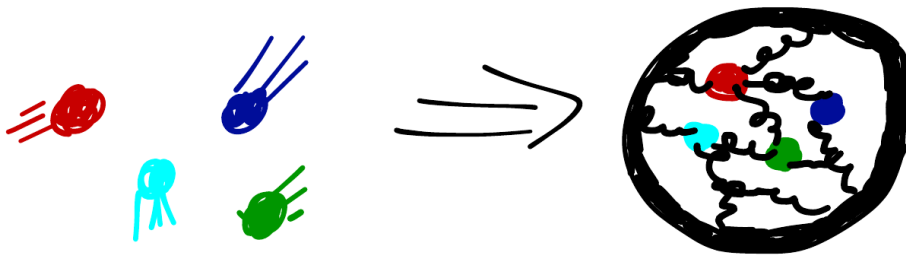
Consistent gravitational evidence  
from kiloparsec to Gpc scales

$$\frac{\Omega_{\text{dark}}}{\Omega_{\text{ordinary}}} \approx 5 \quad \dots \text{not } 10^5 \text{ or } 10^{-5}$$

→ non-gravitational interactions  
with standard model



# Composite dark matter



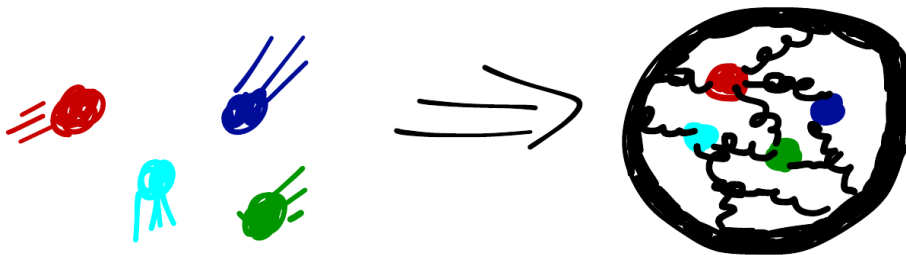
## Early universe

Deconfined charged fermions  $\longrightarrow$  non-gravitational interactions

## Present day

Confined neutral 'dark baryons'  $\longrightarrow$  no experimental detections so far

# Composite dark matter



## Experimental signals

Direct detection and collider searches depend on dark baryon **form factors**

Gravitational waves depend on dark sector **phase transitions**

Need lattice calculations for quantitative predictions

# Lattice Strong Dynamics Collaboration



Argonne Xiao-Yong Jin, James Osborn

Bern Andy Gasbarro

Boston Casey Berger, Rich Brower, Evan Owen, Claudio Rebbi

Colorado Anna Hasenfratz, Ethan Neil, Curtis Peterson

UC Davis Joseph Kiskis

Livermore Dean Howarth, Pavlos Vranas

Liverpool Chris Culver, DS

Michigan Enrico Rinaldi

Nvidia Evan Weinberg

Oregon Graham Kribs

Siegen Oliver Witzel

Trieste James Ingoldby

Yale Thomas Appelquist, Kimmy Cushman, George Fleming

Exploring the range of possible phenomena in strongly coupled field theories

Lattice studies of four-flavour SU(4) dark sector

→ lightest **scalar** ‘baryon’ is stable dark matter candidate

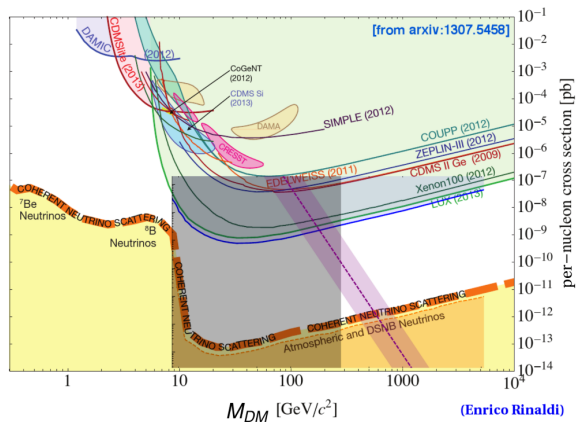
## Direct detection

### Symmetries

→ electric **polarizability**  
is leading interaction

## Collider searches

Charged ‘meson’ Drell–Yan  
rules out shaded region



# Gravitational waves

arXiv:2006.16429

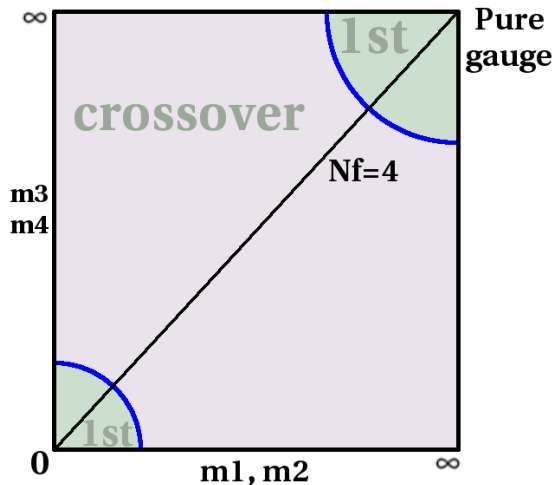
First-order confinement transition  $\longrightarrow$  stochastic background of grav. waves

Pure-gauge transition is first order

Becomes stronger as  $N$  increases

First-order transition persists  
for sufficiently heavy fermions

Four-flavour  $SU(4)$  lattice studies  
 $\longrightarrow$  need  $M_P/M_V \gtrsim 0.9$



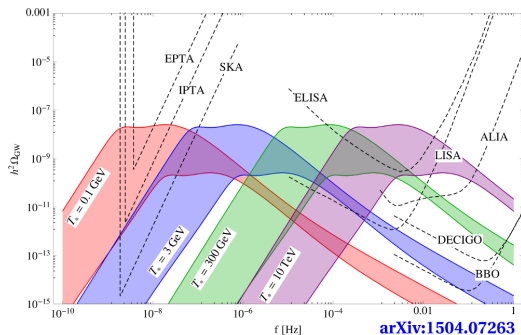


# Promising directions for composite dark matter

First-order transition located  $\longrightarrow$  now need lattice analyses of its properties

Currently computing **latent heat**

Gravitational wave spectrum  
also sensitive to supercooling,  
bubble nucleation rate & wall speed



Another investigation currently underway

Baryon–baryon scattering to explore dark ‘nuclei’ and sub-galactic structure

# Application: Composite Higgs

Large Hadron Collider priority

Study fundamental nature of the Higgs

Composite Higgs sector  
can stabilize electroweak scale

New strong dynamics must differ from QCD  
→ need lattice calculations



# Near-conformality in composite Higgs models

Nearly conformal dynamics makes scale separation natural

→ consistent with non-observation of new particles at LHC

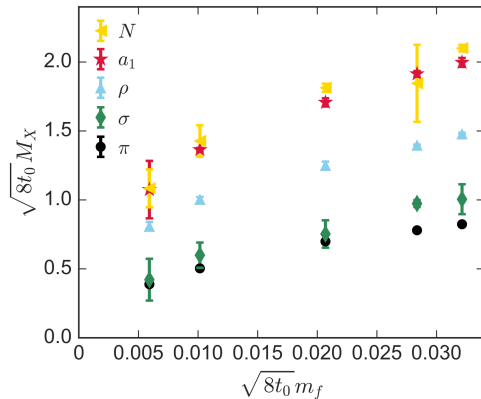
Near-conformal lattice studies

generically observe light scalar

Review: Witzel, [arXiv:1901.08216](https://arxiv.org/abs/1901.08216)

Requires reformulating

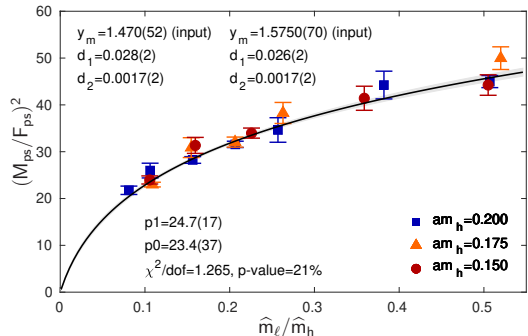
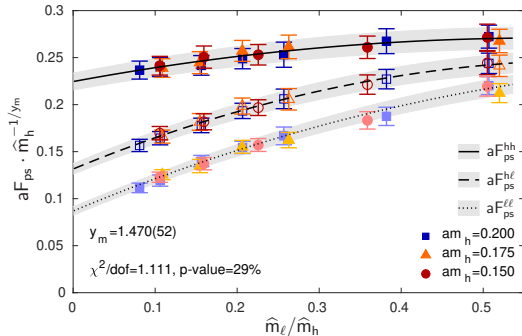
low-energy chiral effective field theory



Build in tunable scale separation

by considering four lighter flavours and six heavier flavours

Results exhibit conformal hyperscaling (**left**) and can be fit to dilaton- $\chi$ PT (**right**)

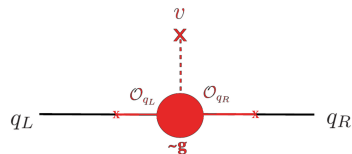
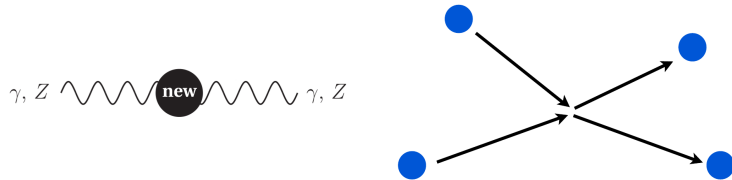


# Promising directions for composite Higgs

## Coming soon to DiRAC

Electroweak  $S$  parameter (vacuum polarization)

$W^\pm W^\pm$  scattering to test effective field theories



## Another investigation being planned

Baryon scaling dimensions for quark & lepton partial compositeness

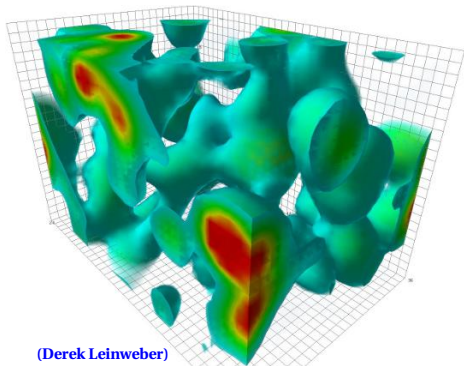
# Application: Supersymmetry and holography

Lattice field theory promises first-principles predictions  
for strongly coupled supersymmetric QFTs

**BSM**

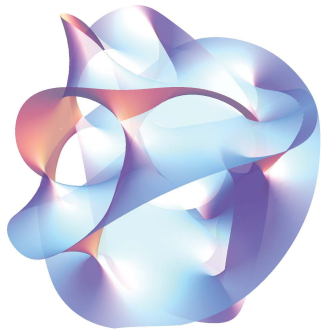


**QFT**



(Derek Leinweber)

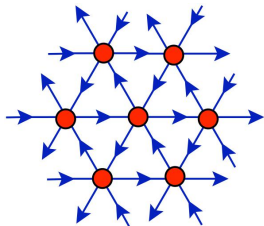
**Holography**



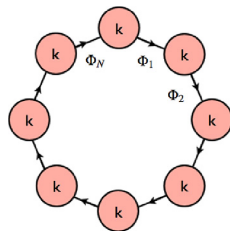
# A brief history of lattice supersymmetry

Supersymmetries 'square' to infinitesimal translations,  $\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\delta^{IJ}\sigma^\mu_{\alpha\dot{\alpha}}P_\mu$   
→ **do not exist** in discrete space-time

**Solution:** Reformulate theory to preserve subset of supersymmetries  
⇒ recover others in continuum limit



Review:  
Catterall–Kaplan–Ünsal,  
[arXiv:0903.4881](https://arxiv.org/abs/0903.4881)



# Testing holographic duality

## Holographic duality conjecture

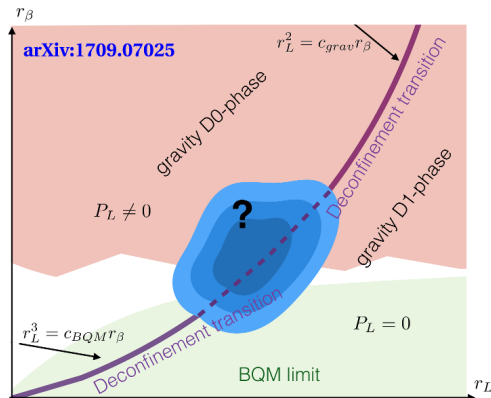
Thermodynamics of supersymmetric QFT  $\longleftrightarrow$  black holes in dual supergravity

2d example: For decreasing  $r_L$   
at low  $t = 1/r_\beta$  and large  $N$

homogeneous black string (D1)  
 $\longrightarrow$  localized black hole (D0)



“spatial deconfinement”  
signalled by Wilson line





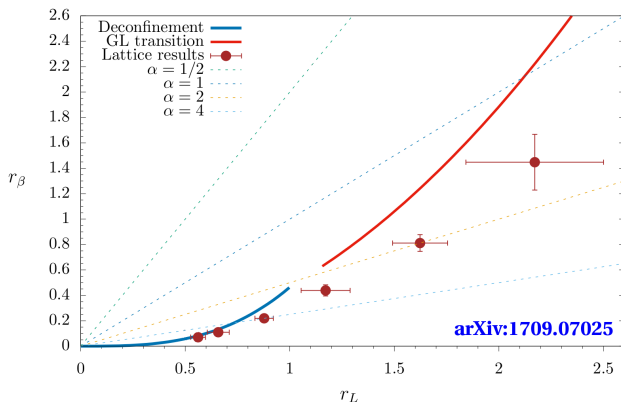
# Two-dimensional super-Yang–Mills phase diagram

Lattice calculations with gauge groups up to SU(16)

Map out transitions in spatial Wilson line

Overall consistent with holography

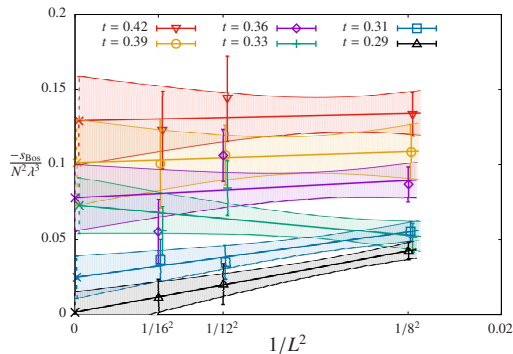
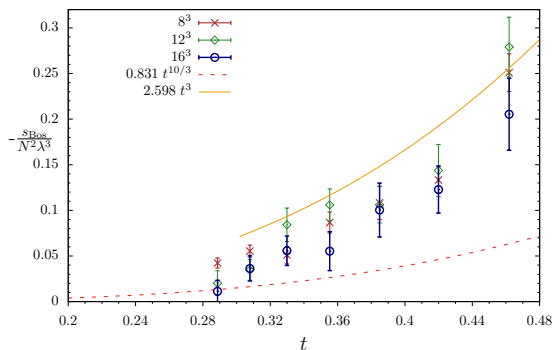
At low temperatures (larger  $r_\beta$ )  
harder to control uncertainties



Moving up to 3d thermodynamics with gauge group SU(8)

For low  $t \lesssim 0.3$  dual black hole energy approaches holographic  $\propto t^{10/3}$

First continuum extrapolations attempted, uncertainties still hard to control

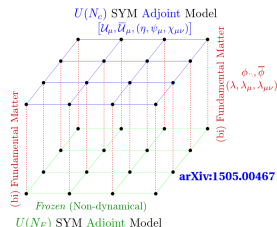
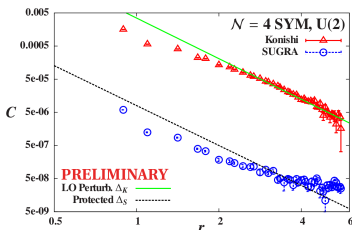
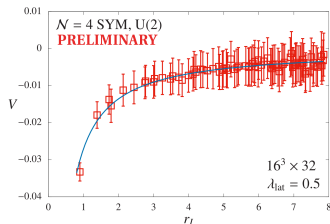


# Promising directions for lattice supersymmetry

## Super-Yang–Mills in four dimensions

The conformal field theory of the original AdS/CFT correspondence

Work underway on static potential, scaling dimensions and more [2102.06775]



## Super-QCD in lower dimensions

Quiver construction preserves subset of supersymmetries for  $d < 4$

# Outlook: Broad applications beckon

Lattice field theory is a broadly applicable tool  
to study strongly coupled quantum field theories

Composite dark matter and gravitational waves

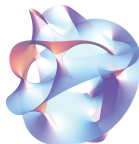
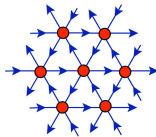
—→ [Colloquium](#) by Kimmy Cushman, 22 April

Composite Higgs and near-conformal dynamics

Supersymmetry and holographic duality

—→ [Longer overview](#), 18 January

—→ [Colloquium](#) by Raghav Govind Jha, 25 February



Thanks for your attention!

Any further questions?

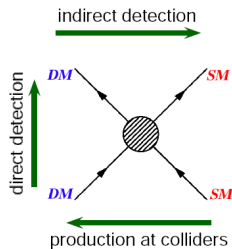
Funding and computing resources

UK Research  
and Innovation



# Backup: Thermal freeze-out for relic density

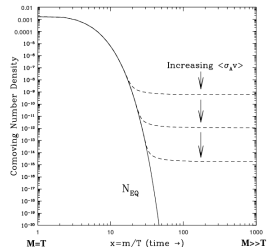
Requires non-gravitational interactions with known particles



$$\text{DM} \longleftrightarrow \text{SM} \text{ for } T \gtrsim M_{DM}$$

$$\text{DM} \longrightarrow \text{SM} \text{ for } T \lesssim M_{DM} \\ \implies \text{rapid depletion of } \Omega_{DM}$$

$$\text{Hubble expansion} \\ \implies \text{dilution} \longrightarrow \text{freeze-out}$$



$$2 \rightarrow 2 \text{ scattering relates coupling and mass, } 200\alpha \sim \frac{M_{DM}}{100 \text{ GeV}}$$

$$\text{Strong } \alpha \sim 16 \longrightarrow \text{'natural' mass scale } M_{DM} \sim 300 \text{ TeV}$$

Smaller  $M_{DM} \gtrsim 1 \text{ TeV}$  possible from  $2 \rightarrow n$  scattering or asymmetry

## Backup: Two roads to natural asymmetric dark matter

**Idea:** Dark matter relic density related to baryon asymmetry

$$\begin{aligned}\Omega_D &\approx 5\Omega_B \\ \implies M_D n_D &\approx 5M_B n_B\end{aligned}$$

$$n_D \sim n_B \implies M_D \sim 5M_B \approx 5 \text{ GeV}$$

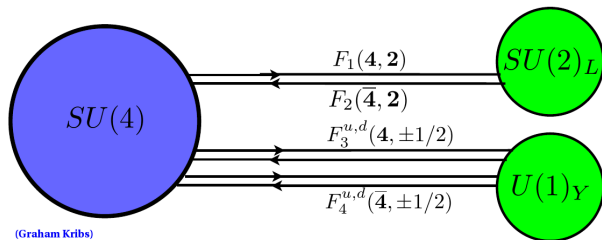
High-dim. interactions relate baryon# and DM# violation

$$M_D \gg M_B \implies n_B \gg n_D \sim \exp[-M_D/T_s] \quad T_s \sim 200 \text{ GeV}$$

EW sphaleron processes above  $T_s$  distribute asymmetries

Both require non-gravitational interactions with known particles

# Backup: More details about SU(4) Stealth Dark Matter



Field	$SU(N_D)$	$(SU(2)_L, Y)$	$Q$
$F_1 = \begin{pmatrix} F_1^u \\ F_1^d \end{pmatrix}$	$\mathbf{N}$	$(\mathbf{2}, 0)$	$\begin{pmatrix} +1/2 \\ -1/2 \end{pmatrix}$
$F_2 = \begin{pmatrix} F_2^u \\ F_2^d \end{pmatrix}$	$\tilde{\mathbf{N}}$	$(\mathbf{2}, 0)$	$\begin{pmatrix} +1/2 \\ -1/2 \end{pmatrix}$
$F_3^u$	$\mathbf{N}$	$(\mathbf{1}, +1/2)$	$+1/2$
$F_3^d$	$\mathbf{N}$	$(\mathbf{1}, -1/2)$	$-1/2$
$F_4^u$	$\tilde{\mathbf{N}}$	$(\mathbf{1}, +1/2)$	$+1/2$
$F_4^d$	$\tilde{\mathbf{N}}$	$(\mathbf{1}, -1/2)$	$-1/2$

Mass terms  $m_V (F_1 F_2 + F_3 F_4) + y (F_1 \cdot H F_4 + F_2 \cdot H^\dagger F_3) + \text{h.c.}$

Vector-like masses evade Higgs-exchange direct detection bounds

Higgs couplings  $\longrightarrow$  charged meson decay before Big Bang nucleosynthesis

**Both required**  $\longrightarrow$  four flavours



# Backup: More details about form factors

## Photon exchange via electromagnetic form factors

Interactions suppressed by powers of confinement scale  $\Lambda \sim M_{DM}$

**Dimension 5:** Magnetic moment  $\rightarrow (\bar{X}\sigma_{\mu\nu}X) F^{\mu\nu}/\Lambda$

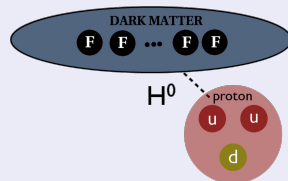
**Dimension 6:** Charge radius  $\rightarrow (\bar{X}X) v_\mu \partial_\nu F^{\mu\nu}/\Lambda^2$

**Dimension 7:** Polarizability  $\rightarrow (\bar{X}X) v_\mu v_\nu F^{\mu\alpha} F_\alpha{}^\nu/\Lambda^3$

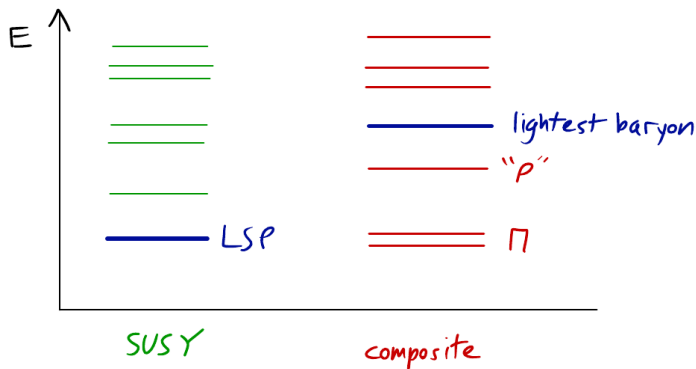
## Higgs exchange via scalar form factors

Higgs couples through  $\sigma$  terms  $\langle B | m_\psi \bar{\psi}\psi | B \rangle$

Produces rapid charged ' $\Pi$ ' decay  
needed for Big Bang nucleosynthesis



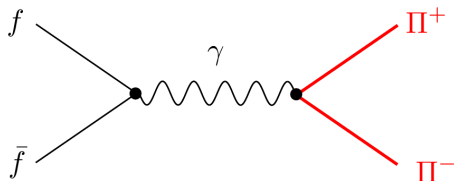
The dark matter is the only stable composite particle, **not** the lightest



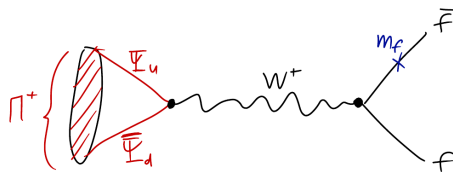
Main constraints from much lighter **charged** “Π”

→ standard ‘missing energy’ searches not efficient

## Production



## Decay



“Particularly tricky” at the LHC: Recent bounds  $M_\Pi \gtrsim 130$  GeV  
 similar to  $M_\Pi \gtrsim 100$  GeV from LEP searches for SUSY tau-partner

Lattice calculation of  $M_{DM}/M_\Pi \longrightarrow M_{DM} \gtrsim 300$  GeV

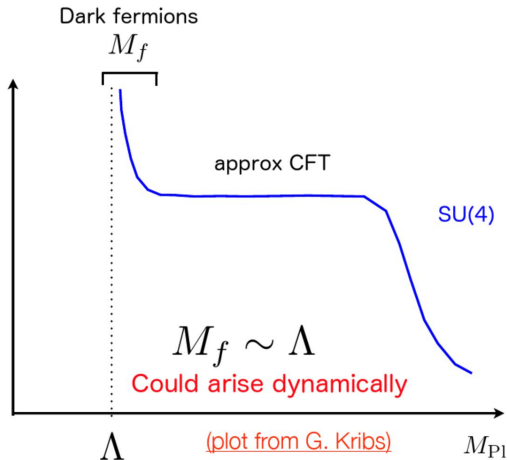
More form factors to compute:  $F_1(4M_\Pi^2)$  for  $\Pi$  and decay constant  $F_\Pi$

## Backup: Stealth Dark Matter mass scales

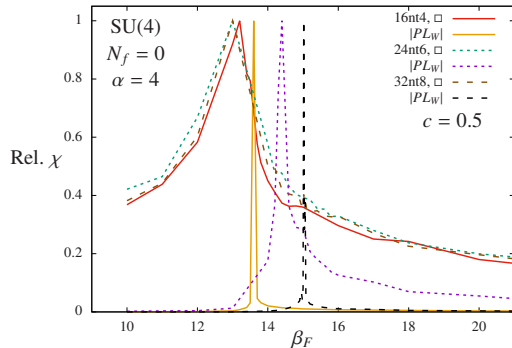
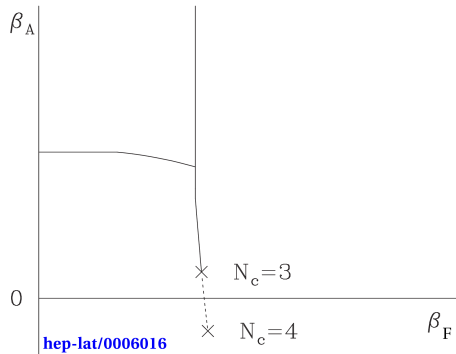
Lattice studies focus on  $m_\psi \simeq \Lambda_{DM}$  where effective theories least reliable

$m_\psi \simeq \Lambda_{DM}$  could arise dynamically

Collider constraints on  $M_{DM}$   
become stronger as  $m_\psi$  decreases



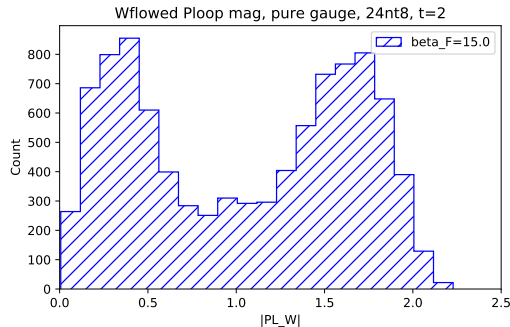
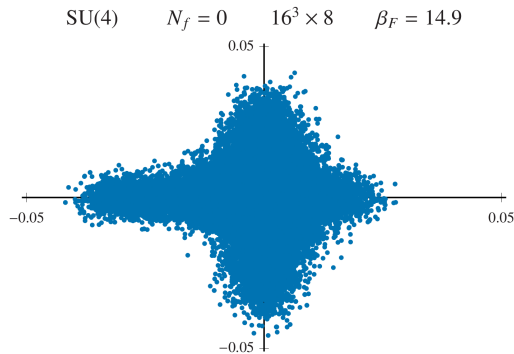
## Backup: Pure gauge checks — Bulk and thermal transitions



Try to avoid bulk transition for small  $N_T \rightarrow$  use  $\beta_A = -\beta_F/4$

Still need  $N_T > 4$  for clear separation between bulk & thermal transitions

# Backup: Pure gauge checks — Order of thermal transition

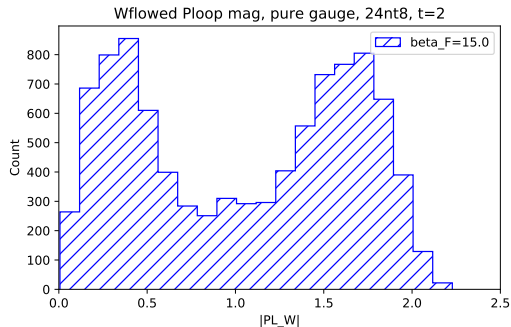


Two peaks in Polyakov loop magnitude histogram  $\rightarrow$  first-order transition ✓

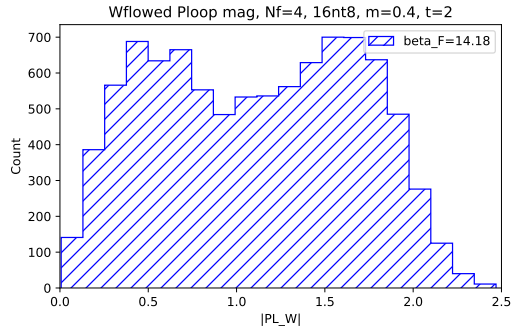
Hysteresis not clearly visible even in pure-gauge case

# Backup: Order of thermal transition with dynamical fermions

## Pure gauge



## Four flavours



Two peaks in Polyakov loop magnitude histogram  $\rightarrow$  first-order transition ✓

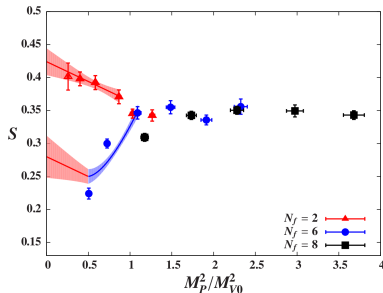
Hysteresis not clearly visible even in pure-gauge case

## Backup: $S$ parameter on the lattice

$$\mathcal{L}_\chi \supset \frac{\alpha_1}{2} g_1 g_2 B_{\mu\nu} \text{Tr} [U_{\tau_3} U^\dagger W^{\mu\nu}] \longrightarrow \gamma, Z \text{ } \text{new} \text{ } \gamma, Z$$

Lattice vacuum polarization calculation provides  $S = -16\pi^2\alpha_1$

Non-zero masses and chiral extrapolation needed due to finite lattice volume



$S = 0.42(2)$  for  $N_F = 2$  matches scaled-up QCD

Larger  $N_F \longrightarrow$  significant reduction

Extrapolation to correct zero-mass limit  
becomes more challenging

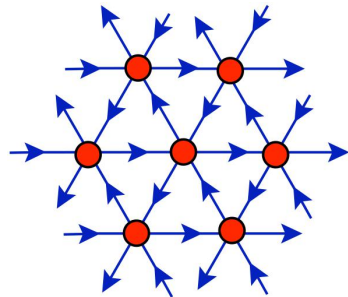


## Backup: Five links in four dimensions $\longrightarrow A_4^*$ lattice

$A_4^* \sim$  4d analog of 2d triangular lattice

Basis vectors linearly dependent and non-orthogonal

Large  $S_5$  point group symmetry



$S_5$  irreps precisely match onto irreps of twisted  $SO(4)_{\text{tw}}$

$$\psi_a \longrightarrow \psi_\mu, \quad \bar{\eta} \quad \text{is} \quad \mathbf{5} \longrightarrow \mathbf{4} \oplus \mathbf{1}$$

$$\chi_{ab} \longrightarrow \chi_{\mu\nu}, \quad \bar{\psi}_\mu \quad \text{is} \quad \mathbf{10} \longrightarrow \mathbf{6} \oplus \mathbf{4}$$

$S_5 \longrightarrow SO(4)_{\text{tw}}$  in continuum limit restores  $Q_a$  and  $Q_{ab}$

## Backup: Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

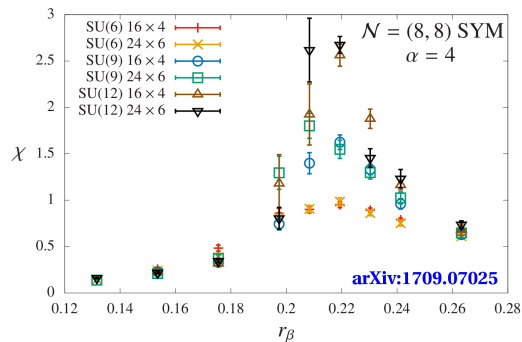
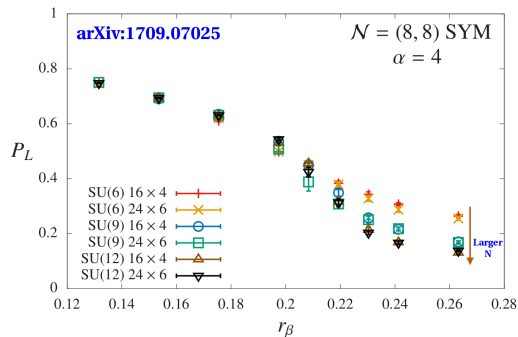
$$\begin{aligned} S_{\text{imp}} &= S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \\ S'_{\text{exact}} &= \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ -\overline{\mathcal{F}}_{ab}(n) \mathcal{F}_{ab}(n) - \chi_{ab}(n) \mathcal{D}_{[a}^{(+)} \psi_{b]}(n) - \eta(n) \overline{\mathcal{D}}_a^{(-)} \psi_a(n) \right. \\ &\quad \left. + \frac{1}{2} \left( \overline{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) + G \sum_{a \neq b} (\det \mathcal{P}_{ab}(n) - 1) \mathbb{I}_N \right)^2 \right] - S_{\text{det}} \\ S_{\text{det}} &= \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr} [\eta(n)] \sum_{a \neq b} [\det \mathcal{P}_{ab}(n)] \text{Tr} [\mathcal{U}_b^{-1}(n) \psi_b(n) + \mathcal{U}_a^{-1}(n + \hat{\mu}_b) \psi_a(n + \hat{\mu}_b)] \\ S_{\text{closed}} &= -\frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ \epsilon_{abcde} \chi_{de}(n + \hat{\mu}_a + \hat{\mu}_b + \hat{\mu}_c) \overline{\mathcal{D}}_c^{(-)} \chi_{ab}(n) \right], \\ S'_{\text{soft}} &= \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_n \sum_a \left( \frac{1}{N} \text{Tr} [\mathcal{U}_a(n) \overline{\mathcal{U}}_a(n)] - 1 \right)^2 \end{aligned} \tag{18}$$

$\gtrsim 100$  inter-node data transfers in the fermion operator — non-trivial...

Public parallel code to reduce barriers to entry: [github.com/daschaich/susy](https://github.com/daschaich/susy)

Evolved from MILC QCD code, user guide in [arXiv:1410.6971](https://arxiv.org/abs/1410.6971)

# Backup: Spatial deconfinement transition signals

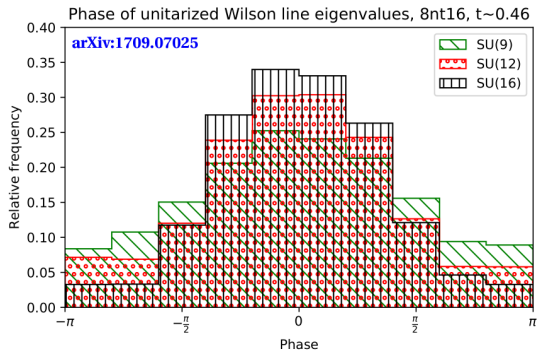
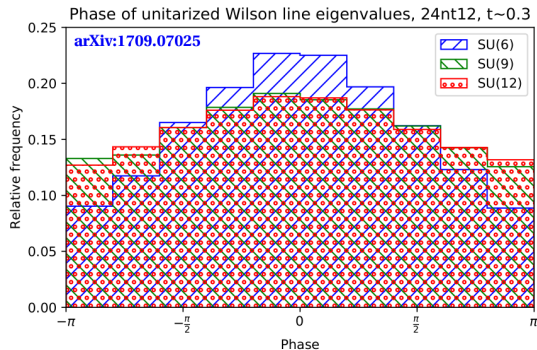


Peaks in Wilson line susceptibility match change in its magnitude  $|PL|$ ,  
grow with size of  $SU(N)$  gauge group, comparing  $N = 6, 9, 12$

Agreement for  $16 \times 4$  vs.  $24 \times 6$  lattices (aspect ratio  $\alpha = r_L/r_\beta = 4$ )

# Backup: 2d $\mathcal{N} = (8, 8)$ SYM Wilson line eigenvalues

Check 'spatial deconfinement' through Wilson line eigenvalue phases

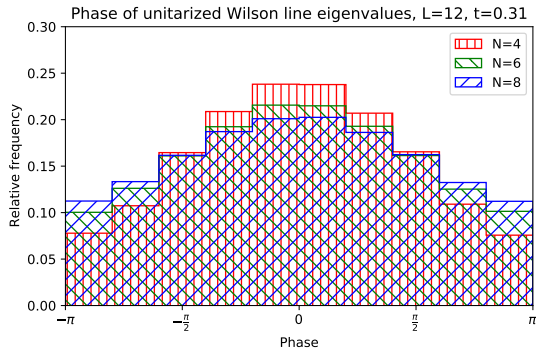
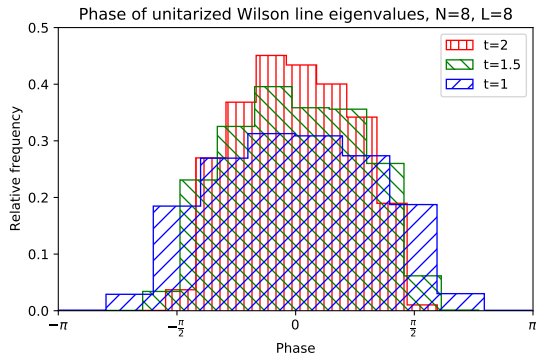


**Left:**  $\alpha = 2$  distributions more extended as  $N$  increases  $\rightarrow$  D1 black string

**Right:**  $\alpha = 1/2$  distributions more compact as  $N$  increases  $\rightarrow$  D0 black hole

## Backup: 3d $\mathcal{N} = 8$ SYM Wilson line eigenvalues

Check 'spatial deconfinement' through Wilson line eigenvalue phases



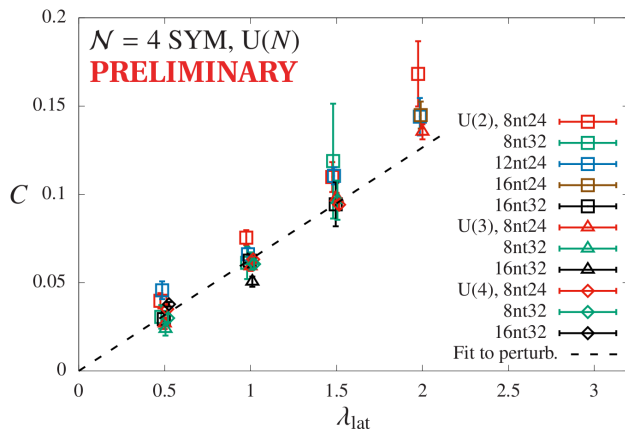
**Left:** High-temperature  $U(8)$   $8^3$  distributions more compact as  $t$  increases

**Right:** Low-temperature  $U(N)$   $12^3$  distributions more uniform as  $N$  increases

## Backup: Coupling dependence of Coulomb coefficient

Continuum perturbation theory  $\longrightarrow C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

Holography  $\longrightarrow C(\lambda) \propto \sqrt{\lambda}$  for  $N \rightarrow \infty$  and  $\lambda \rightarrow \infty$  with  $\lambda \ll N$



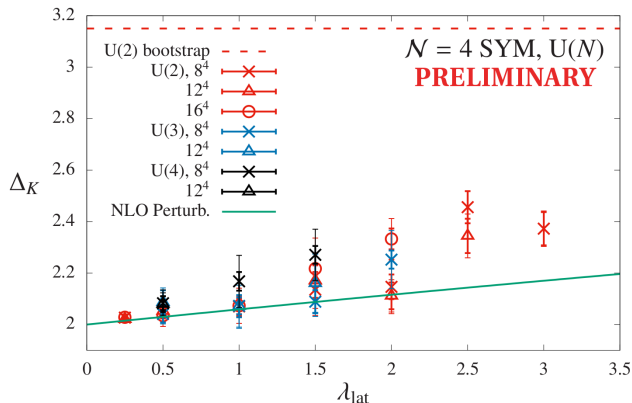
For  $\lambda_{\text{lat}} \leq 2$ , consistent with  
leading-order perturbation theory

## Backup: Preliminary $\Delta_K$ results from Monte Carlo RG

Analyzing both  $\mathcal{O}_K^{\text{lat}}$  and  $\mathcal{O}_S^{\text{lat}}$

Imposing protected  $\Delta_S = 2$   
 $\longrightarrow \Delta_K(\lambda)$  looks perturbative

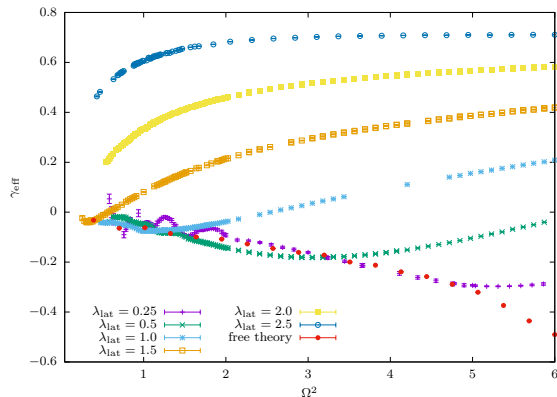
Systematic uncertainties from  
different amounts of smearing



Complication from twisting  $SO(4)_R \subset SO(6)_R$

$\mathcal{O}_K^{\text{lat}}$  mixes with  $SO(4)_R$ -singlet part of  $SO(6)_R$ -nonsinglet  $\mathcal{O}_S$   
 $\longrightarrow$  disentangle via variational analyses

Fermion op. eigenvalues predict 'mass' anomalous dimension of fermion bilinear  
Should vanish  $\rightarrow$  test discretization and finite-volume effects in lattice calcs



Scale-dependent 'effective anom. dim.'  
due to broken conformality

Recover true critical exponent  
at low energy scale  $\Omega^2 \ll 1$

$0.25 \leq \lambda_{\text{lat}} \leq 2.5$ , with even **free theory**  
sensitive to lattice effects

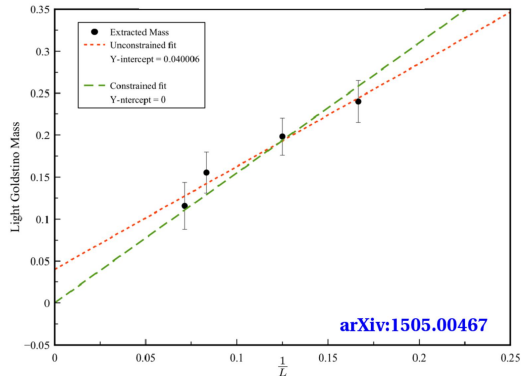
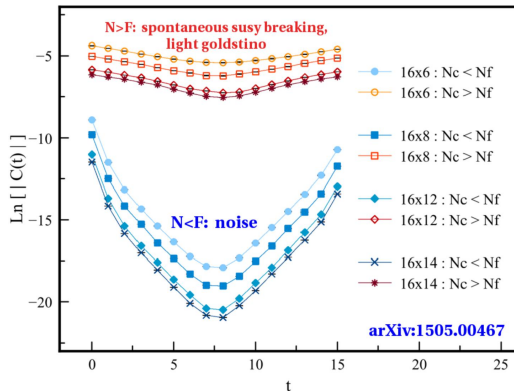


# Backup: Dynamical susy breaking in 2d lattice superQCD

$U(N)$  superQCD with  $F$  fundamental hypermultiplets

Observe spontaneous susy breaking only for  $N > F$ , as expected

Catterall–Veernala, [arXiv:1505.00467](https://arxiv.org/abs/1505.00467)



## Backup: More on dynamical susy breaking

Spontaneous susy breaking means  $\langle 0 | H | 0 \rangle > 0$  or equivalently  $\langle \mathcal{Q}\mathcal{O} \rangle \neq 0$

Twisted superQCD auxiliary field e.o.m.  $\longleftrightarrow$  Fayet–Iliopoulos  $D$ -term potential

$$d = \overline{\mathcal{D}}_a \mathcal{U}_a + \sum_{i=1}^F \phi_i \overline{\phi}_i - r \mathbb{I}_N \quad \longleftrightarrow \quad \text{Tr} \left[ \left( \sum_i \phi_i \overline{\phi}_i - r \mathbb{I}_N \right)^2 \right] \in H$$

Have  $F \times N$  scalar vevs to zero out  $N \times N$  matrix

$\longrightarrow N > F$  suggests susy breaking,  $\langle 0 | H | 0 \rangle > 0 \longleftrightarrow \langle \mathcal{Q}\eta \rangle = \langle d \rangle \neq 0$