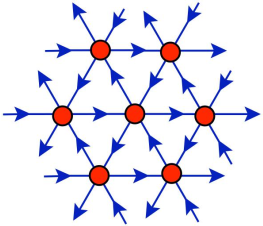
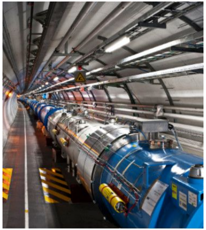
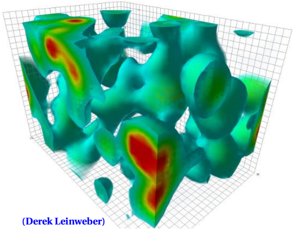
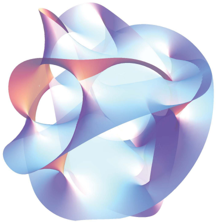
Progress and prospects of lattice supersymmetry

David Schaich (University of Liverpool)



Nonperturbative and Numerical Approaches
to Quantum Gravity, String Theory and Holography

International Centre for Theoretical Sciences, Bangalore, 18 January 2021

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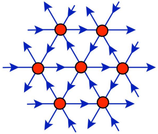
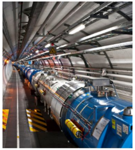
Overview and plan

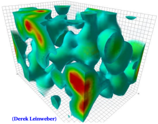


Significant progress currently being made
in lattice studies of supersymmetric QFTs



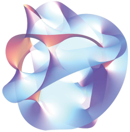
Motivation and background

Special cases: $\mathcal{N} = 1$ super-Yang–Mills; Matrix models



Lattice $\mathcal{N} = 4$ super-Yang–Mills

- Formulation highlights
- Dimensionally reduced (2d & 3d) thermodynamics
- 4d static potential & scaling dimensions



Prospects and future directions

Overview and plan

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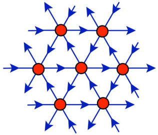

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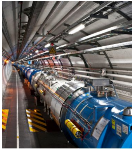

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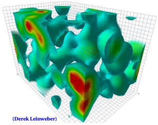
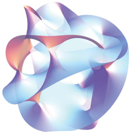


Prospects and future directions

These slides: davidschaich.net/talks/2101Bangalore.pdf

Interaction encouraged — complete coverage unnecessary

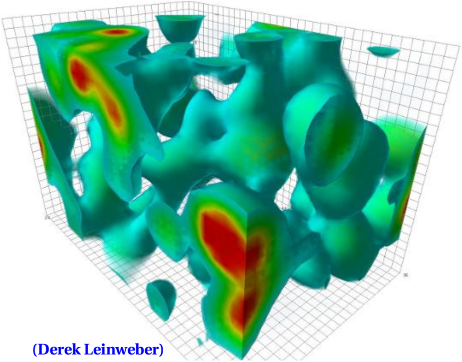


Motivations

Lattice field theory promises first-principles predictions
for strongly coupled supersymmetric QFTs

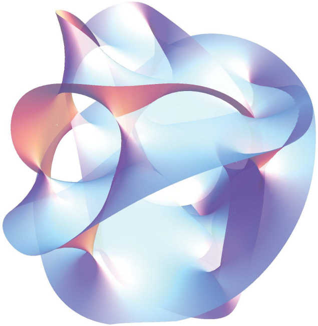
BSM

QFT



(Derek Leinweber)

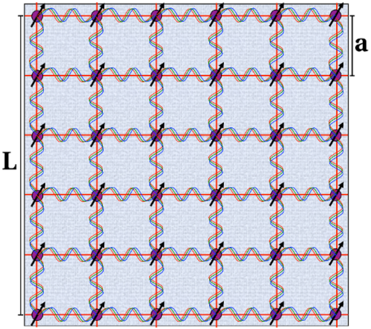
Holography



Background: Lattice field theory in a nutshell

Formally $\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \;\; \mathcal{O}(\Phi) \;\; e^{-S[\Phi]}$

Regularize by formulating theory in finite, discrete, euclidean space-time



P. Vranas LLNL

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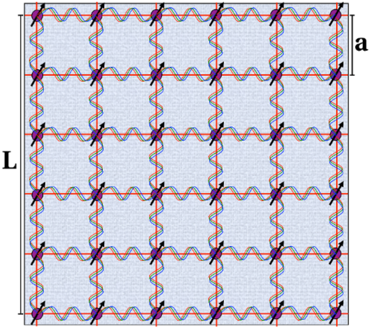
Formally $\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}}\int\mathcal{D}\Phi\ \mathcal{O}(\Phi)\ e^{-S[\Phi]}$

Regularize by formulating theory in finite, discrete, euclidean space-time
↖Gauge invariant, non-perturbative, d-dimensional



P. Vranas LLNL

Spacing between lattice sites ("a")
$\longrightarrow$ UV cutoff scale $1/a$

Remove cutoff: $a\to 0$ $(L/a\to\infty)$

Discrete $\longrightarrow$ continuous symmetries ✓

Numerical lattice field theory calculations

High-performance computing
$\longrightarrow$ evaluate up to
$\sim$billion-dimensional integrals

(Dirac op. as $\sim 10^9 \times 10^9$ matrix)

Importance sampling Monte Carlo

Algorithms sample field configurations with probability $\frac{1}{\mathcal{Z}} e^{-S[\Phi]}$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \;\; \mathcal{O}(\Phi) \;\; e^{-S[\Phi]} \longrightarrow \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(\Phi_i) \text{ with stat. uncertainty } \propto \frac{1}{\sqrt{N}}$$

Supersymmetry must be broken on the lattice

Supersymmetry is a space-time symmetry, $(\mathrm{I} = 1, \cdots, \mathcal{N})$
adding spinor generators Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ to translations, rotations, boosts

$\left\{Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{J}}_{\dot{\alpha}}\right\} = 2\delta^{\mathrm{IJ}}\sigma^{\mu}_{\alpha\dot{\alpha}}P_{\mu}$ broken in discrete space-time
$\longrightarrow$ relevant susy-violating operators

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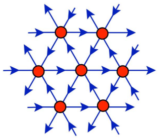
$\longrightarrow$ relevant susy-violating operators

Simplifications have helped enable significant recent progress

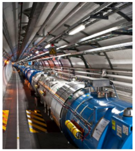
Avoid scalars | **Reduce dimensions** | **Maximize symmetries**

Checkpoint

Significant progress currently being made
in lattice studies of supersymmetric QFTs

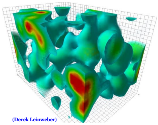


✓ Motivation and background — Questions?



Special cases: $\mathcal{N} = 1$ super-Yang–Mills; Matrix models

Lattice $\mathcal{N} = 4$ super-Yang–Mills



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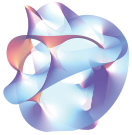


Prospects and future directions

Avoid scalars: $\mathcal{N} = 1$ super-Yang–Mills

SU(N) gauge theory with single massless Majorana fermion in adjoint rep.

Straightforward lattice fermion formulations explicitly break chiral symmetry
$\longrightarrow$ large additive gluino mass renormalization

Chiral ('domain wall' or 'overlap') lattice fermions numerically expensive

Avoid scalars: $\mathcal{N} = 1$ super-Yang–Mills

SU(N) gauge theory with single massless Majorana fermion in adjoint rep.

1) Fine-tune gluino mass $\longrightarrow$ supersymmetry in chiral continuum limit

2) Domain wall or overlap fermions

$\longrightarrow$ automatic (accidental) supersymmetry in continuum limit

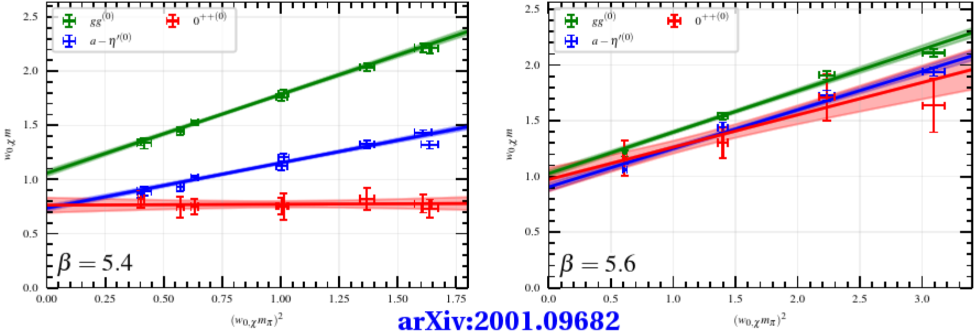
Selected recent progress fine-tuning gluino mass

Scalar, pseudoscalar and fermionic partner

approach degenerate supermultiplet for massless gluino

Smaller lattice spacing 'a' (larger β) $\longrightarrow$ improved supermultiplet formation

Desy–Münster–Regensburg–Jena, arXiv:2001.09682



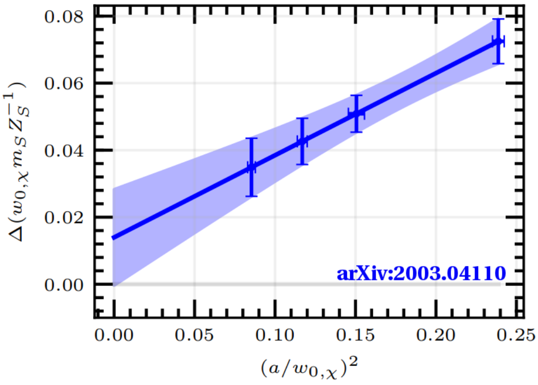
arXiv:2001.09682

Selected recent progress fine-tuning gluino mass

Measure of supersymmetry breaking from Ward identities

vanishes in chiral continuum limit, $a^2 \to 0$

Desy–Münster–Regensburg–Jena, arXiv:2003.04110



Extrapolation consistent

with $\mathcal{O}(a^2)$ discretization artifacts

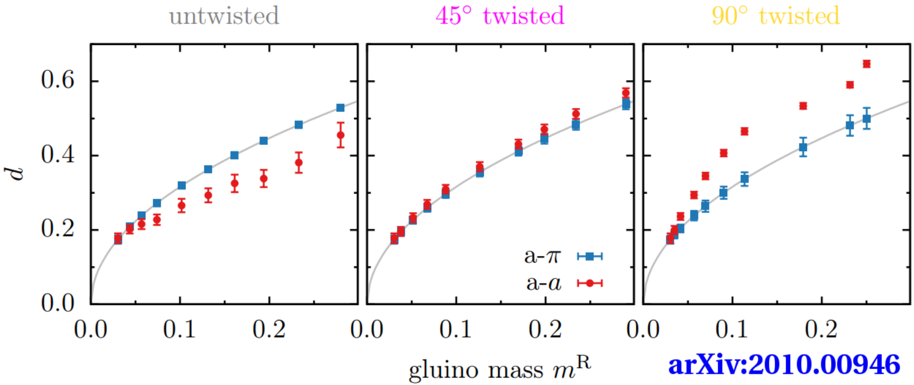
expected for this lattice action

Selected recent progress fine-tuning gluino mass

Alternate 'twisted-mass' action provides extra 'twist angle' parameter

$\longrightarrow$ tune this to improve approach to continuum limit

Steinhauser–Sternbeck–Wellegehausen–Wipf, arXiv:2010.00946

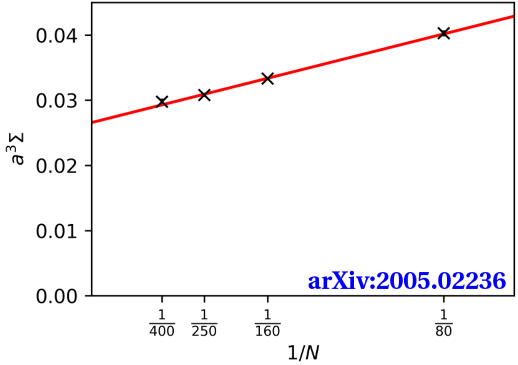


arXiv:2010.00946

Recent progress with overlap $\mathcal{N} = 1$ super-Yang–Mills

N-order polynomial approximation to compute $\text{sign}(D) = \dfrac{D}{\sqrt{D \cdot D}}$ for large matrix

Piemonte–Bergner–López, arXiv:2005.02236



Bare gluino condensate from 12^4 lattices

$N < \infty \quad \longrightarrow \quad$ non-zero 'residual mass'

but only multiplicative renormalization

Reduce dimensions: Supersymmetric quantum mechanics

Ultimate simplification — compactify all spatial dimensions

4d SU(N) super-Yang–Mills (SYM) $\longrightarrow$ quantum mechanics of $N \times N$ matrices

Both lattice and non-lattice numerical approaches viable

More on this topic from Jun Nishimura tomorrow

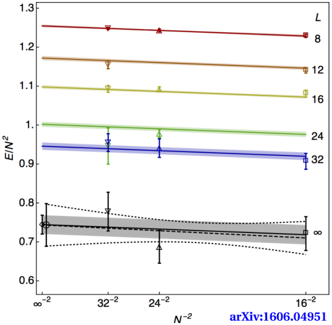
16-supercharge theories motivated by holography

Thermodynamics of maximal SYM $\longleftrightarrow$ black holes in string theory

Testing holography with lattice super-Yang–Mills QM

Predict corrections to SUGRA result through large-N continuum extrapolations

Monte Carlo String/M-Theory Collaboration, arXiv:1606.04951



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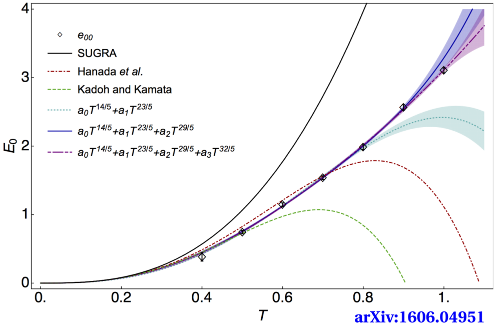


$16 \leq N \leq 32$ — MMMM code parallelizes individual $N \times N$ matrices

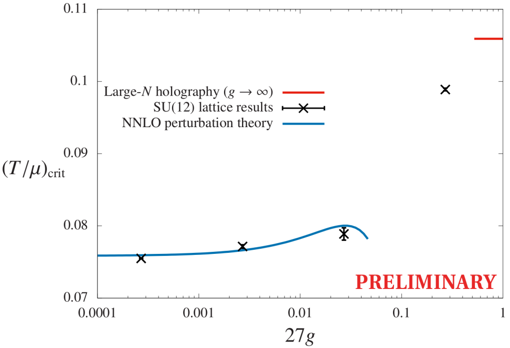
Recent progress: Supersymmetric mass deformation

Berenstein–Maldacena–Nastase, hep-th/0202021

Deform SYM QM while preserving maximal supersymmetry

$\longrightarrow$ more interesting features including phase transition at critical T/mu

DS–Jha–Joseph, 2003.01298 & to appear



Phase diagram of critical T/μ

vs. dimensionless coupling g

For small $g \lesssim 10^{-3}$, agree with

NNLO perturbation theory

Approach leading-order holography

as g increases

Checkpoint

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in lattice studies of supersymmetric QFTs

✓ Motivation and background

✓ Special cases: $\mathcal{N} = 1$ SYM; Matrix models — Questions?

Lattice $\mathcal{N} = 4$ super-Yang–Mills

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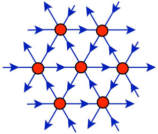
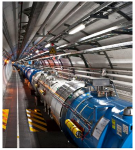
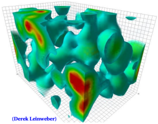
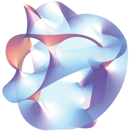
Prospects and future directions







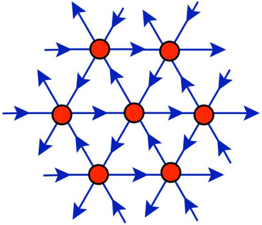


Maximize symmetries: Lattice $\mathcal{N} = 4$ super-Yang–Mills

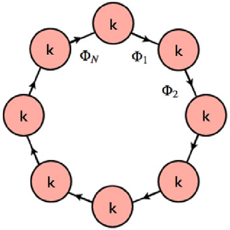
Preserve susy sub-algebra in discrete lattice space-time

$\Longrightarrow$ correct continuum limit with little or no fine tuning

Equivalent constructions from 'topological' twisting and dim'l deconstruction



Review:
Catterall–Kaplan–Ünsal,
arXiv:0903.4881



Need 2^d supersymmetries in d dimensions

$d = 4 \longrightarrow \mathcal{N} = 4$ super-Yang–Mills (SYM)

$\mathcal{N} = 4$ SYM in a nutshell

Arguably simplest non-trivial 4d QFT $\longrightarrow$ dualities, amplitudes, ...

SU(N) gauge theory with $\mathcal{N} = 4$ fermions Ψ^{I} and 6 scalars Φ^{IJ},
all massless and in adjoint rep.

Symmetries relate coefficients of kinetic, Yukawa and Φ^4 terms

Maximal 16 supersymmetries Q_α^{I} and $\overline{Q}_{\dot\alpha}^{\mathrm{I}}$ $\quad \mathrm{I} = 1, \cdots, 4$
transform under global SU(4) $\sim$ SO(6) R symmetry

Conformal $\longrightarrow$ β function is zero for all values of $\lambda = g^2 N$

Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}} \gamma_5$$

$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b \qquad \text{with } a, b = 1, \cdots, 5$$

R-symmetry index $\times$ Lorentz index $\Longrightarrow$ reps of 'twisted rotation group'

$$\text{SO}(4)_{\text{tw}} \equiv \text{diag}\Big[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R\Big] \qquad\qquad \text{SO}(4)_R \subset \text{SO}(6)_R$$

Change of variables $\longrightarrow$ $\mathcal{Q}$s transform with integer 'spin' under $\text{SO}(4)_{\text{tw}}$

Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}} \gamma_5$$

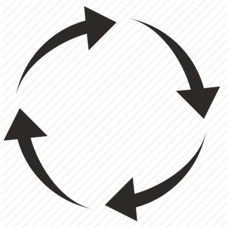
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$

with $a, b = 1, \cdots, 5$

Discrete space-time

Can preserve closed sub-algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$



Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}} \gamma_5$$

$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$

with $a, b = 1, \cdots, 5$

Discrete space-time

Can preserve closed sub-algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$

Completing the twist

Fields also transform with integer spin under $\mathrm{SO}(4)_{\mathrm{tw}}$ — no spinors

$$\Psi \text{ and } \overline{\Psi} \longrightarrow \eta,\ \psi_a \text{ and } \chi_{ab}$$

$$A_\mu \text{ and } \Phi^{\mathrm{I}} \longrightarrow \text{complexified gauge field } \mathcal{A}_a \text{ and } \overline{\mathcal{A}}_a$$

$$\longrightarrow \mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1) \text{ gauge theory}$$

✓ $\mathcal{Q}$ interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\mathcal{Q}\,\mathcal{A}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

$\nwarrow$ bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Covariant derivatives $\longrightarrow$ finite difference operators

Complexified gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $\mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C})$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

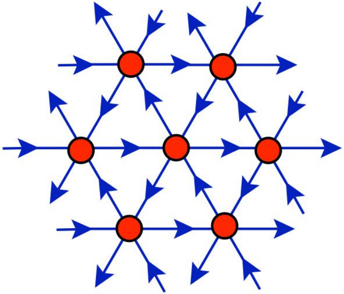
Geometry: η on sites, ψ_a on links, etc.

Supersymmetric lattice action $(\mathcal{Q}S = 0)$ from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity

$$S = \frac{N}{4\lambda_{\text{lat}}} \text{Tr}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de}\right]$$

Five links in four dimensions $\longrightarrow$ A_4^* lattice

$A_4^* \sim$ 4d analog of 2d triangular lattice



Basis vectors linearly dependent and non-orthogonal

Large S_5 point group symmetry

S_5 irreps precisely match onto irreps of twisted $\text{SO}(4)_{\text{tw}}$

$$\begin{array}{lll} \psi_a \longrightarrow \psi_\mu, \ \ \overline{\eta} & \text{is} & \mathbf{5} \longrightarrow \mathbf{4} \oplus \mathbf{1} \\ \chi_{ab} \longrightarrow \chi_{\mu\nu}, \ \ \overline{\psi}_\mu & \text{is} & \mathbf{10} \longrightarrow \mathbf{6} \oplus \mathbf{4} \end{array}$$

$S_5 \longrightarrow \text{SO}(4)_{\text{tw}}$ in continuum limit restores $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Formal formulation features

Analytic results for twisted $\mathcal{N} = 4$ SYM on A_4^* lattice

U(N) gauge invariance + $\mathcal{Q}$ + S_5 lattice symmetries

$\longrightarrow$ Moduli space preserved to all orders

$\longrightarrow$ One-loop lattice β function vanishes

$\longrightarrow$ Only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

[arXiv:1102.1725, arXiv:1306.3891, arXiv:1408.7067]

Not yet practical for numerical calculations

Must regulate zero modes and flat directions, especially in U(1) sector

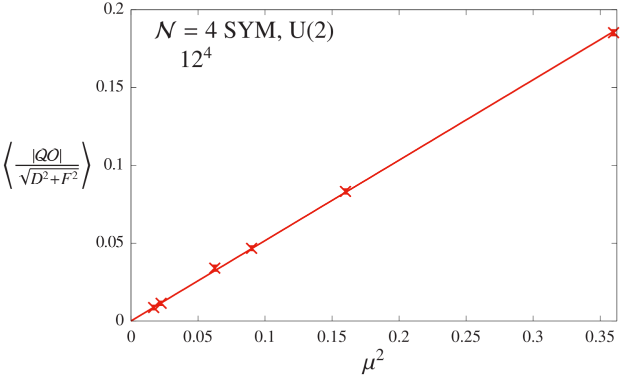
Two deformations stabilize lattice calculations

1) Add SU(N) scalar potential $\propto \mu^2 \sum_a \left(\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - N\right)^2$

Softly breaks susy $\longrightarrow$ $\mathcal{Q}$-violating operators vanish $\propto \mu^2 \to 0$

Test via Ward identity violations

$$\mathcal{Q}\left[\eta\mathcal{U}_a\overline{\mathcal{U}}_a\right] \neq 0$$



Two deformations stabilize lattice calculations

2) Constrain U(1) plaquette determinant $\sim G \sum_{a<b} \left(\det \mathcal{P}_{ab} - 1\right)$

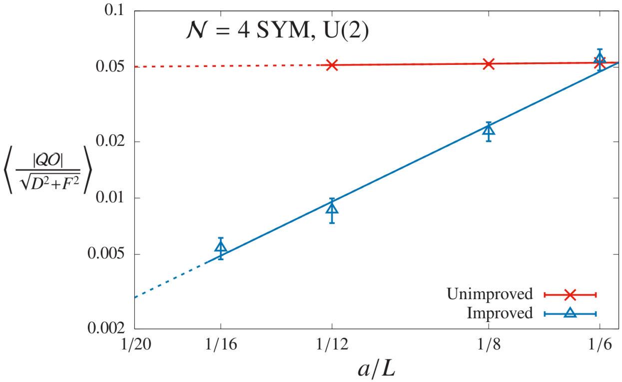
Implemented supersymmetrically as Fayet–Iliopoulos D-term potential

Test via Ward identity violations

$$\mathcal{Q}\left[\eta \mathcal{U}_a \overline{\mathcal{U}}_a\right] \neq 0$$

Log–log axes

$\longrightarrow$ violations $\propto (a/L)^2$



Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

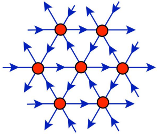
$$\begin{aligned}
S_{\text{imp}} &= S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \qquad (18)\\
S'_{\text{exact}} &= \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr}\left[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_a\psi_a(n) \right.\\
&\qquad \left. + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a(n) + G\sum_{a\neq b}\left(\det\mathcal{P}_{ab}(n) - 1\right)\mathbb{I}_N\right)^2 \right] - S_{\text{det}}\\
S_{\text{det}} &= \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr}\left[\eta(n)\right] \sum_{a\neq b} \left[\det\mathcal{P}_{ab}(n)\right] \text{Tr}\left[\mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n+\widehat{\mu}_b)\psi_a(n+\widehat{\mu}_b)\right]\\
S_{\text{closed}} &= -\frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr}\left[\epsilon_{abcde}\ \chi_{de}(n+\widehat{\mu}_a+\widehat{\mu}_b+\widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_c\chi_{ab}(n)\right],\\
S'_{\text{soft}} &= \frac{N}{4\lambda_{\text{lat}}}\mu^2 \sum_n\sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right)^2
\end{aligned}$$

$\gtrsim$100 inter-node data transfers in the fermion operator — non-trivial. . .

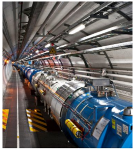
Public parallel code to reduce barriers to entry: github.com/daschaich/susy

Evolved from MILC QCD code, user guide in arXiv:1410.6971

Checkpoint

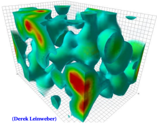


Significant progress currently being made
in lattice studies of supersymmetric QFTs



✓ Motivation and background

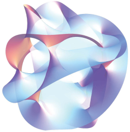
✓ Special cases: $\mathcal{N} = 1$ super-Yang–Mills; Matrix models



Lattice $\mathcal{N} = 4$ super-Yang–Mills

✓ Formulation highlights — Questions?

- Dimensionally reduced (2d & 3d) thermodynamics
- 4d static potential & scaling dimensions



Prospects and future directions

2d thermodynamics on $(r_L \times r_\beta)$ torus

arXiv:1709.07025

Dimensionally reduce to (deconfined) 2d $\mathcal{N} = (8, 8)$ SYM with four scalar $\mathcal{Q}$

Low temperatures $t = 1/r_\beta \longleftrightarrow$ black holes in dual supergravity

For decreasing r_L at large N

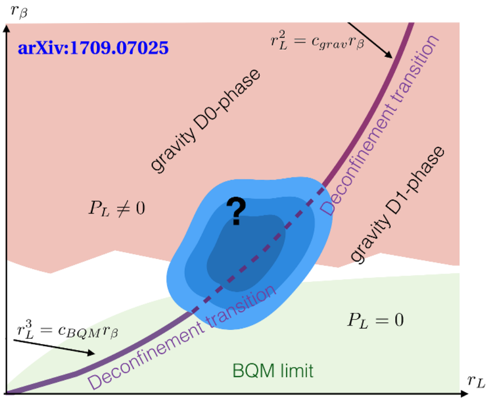
homogeneous black string (D1)

$\longrightarrow$ localized black hole (D0)

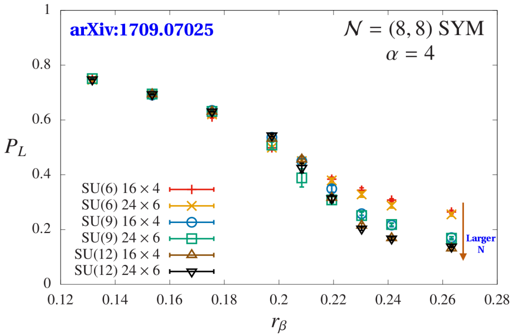
$\updownarrow$

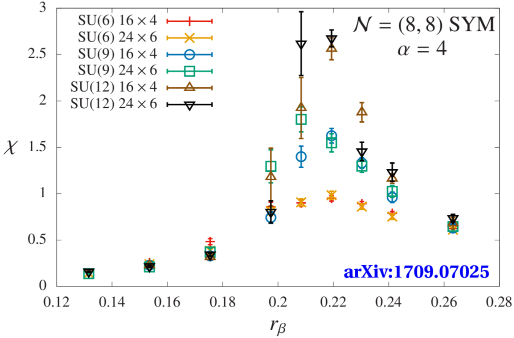
"spatial deconfinement"

signalled by Wilson line P_L



Spatial deconfinement transition signals





Peaks in Wilson line susceptibility match change in its magnitude |PL|,

grow with size of SU(N) gauge group, comparing $N = 6, 9, 12$

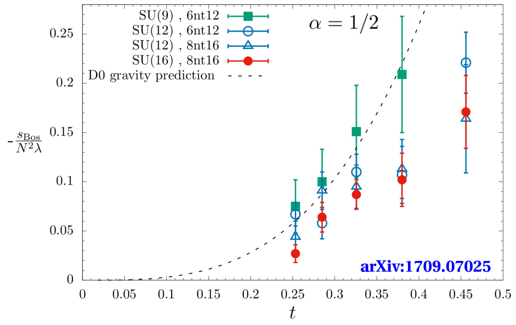
Agreement for 16×4 vs. 24×6 lattices (aspect ratio $\alpha = r_L / r_\beta = 4$)

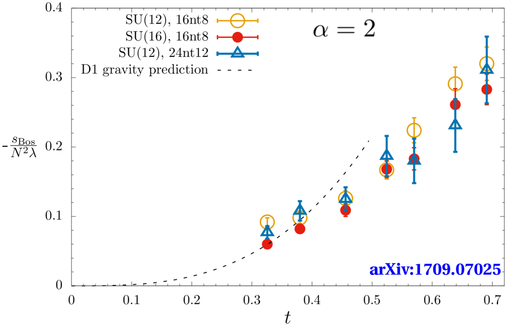
Check holographic black hole energies

Lattice results consistent with leading expectation for sufficiently low $t \lesssim 0.4$

Similar behavior $\longrightarrow$ difficult to distinguish phases

$\propto t^{3.2}$ for small-r_L D0 phase $\propto t^3$ for large-r_L D1 phase

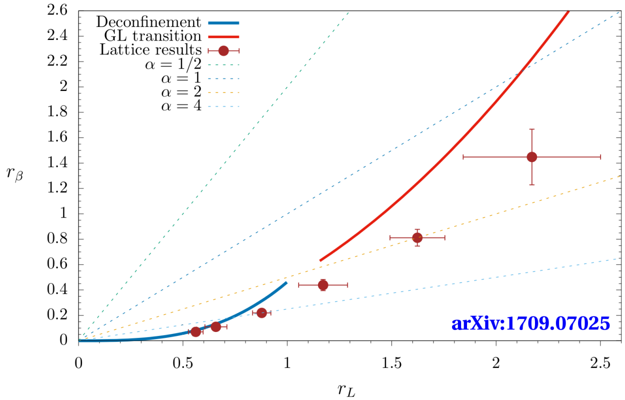




Lattice results for 2d $\mathcal{N} = (8,8)$ SYM phase diagram

Good agreement with bosonic QM at high temperatures

Harder to control low-temperature uncertainties (larger $N > 16$ should help)



Overall consistent with holography

Comparing multiple lattice sizes and $6 \leq N \leq 16$

Controlled extrapolations are work in progress

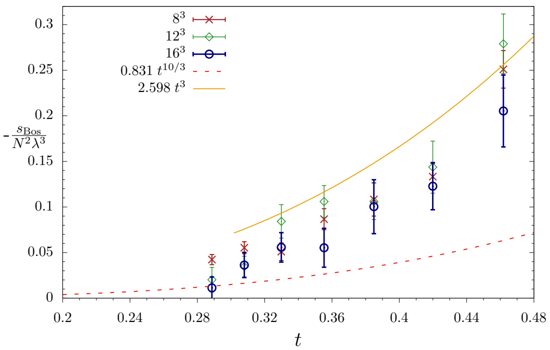
3d thermodynamics and continuum extrapolation

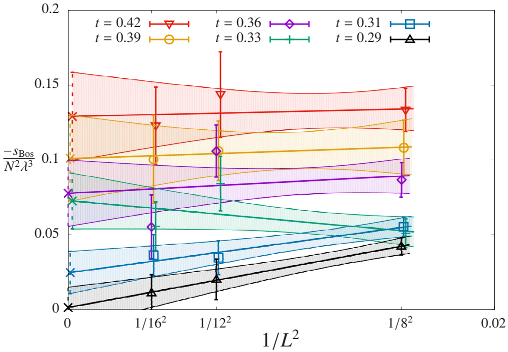
arXiv:2010.00026

Similar dimensional reduction to 3d $\mathcal{N} = 8$ SYM with two scalar $\mathcal{Q}$

Again approach leading holographic expectation $\propto t^{10/3}$ for low $t \lesssim 0.3$

Carry out continuum extrapolations for fixed aspect ratio $\alpha = 1$ and $N = 8$





Checkpoint

Significant progress currently being made in lattice studies of supersymmetric QFTs

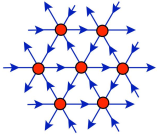
✓ Motivation and background

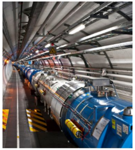
✓ Special cases: $\mathcal{N} = 1$ super-Yang–Mills; Matrix models

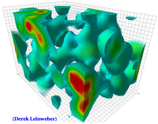
Lattice $\mathcal{N} = 4$ super-Yang–Mills

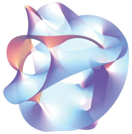
- ✓ Formulation highlights
- ✓ Dimensionally reduced thermodynamics — Questions?
- 4d static potential & scaling dimensions

Prospects and future directions





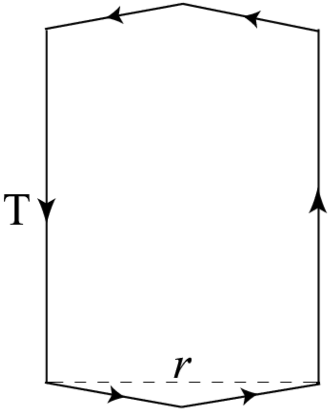




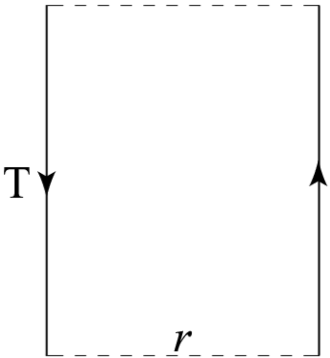
Static potential $V(r)$ for 4d $\mathcal{N} = 4$ SYM

Static probes $\longrightarrow$ $r \times T$ Wilson loops $W(r, T) \propto e^{-V(r)\,T}$

Coulomb gauge trick reduces A_4^* lattice complications



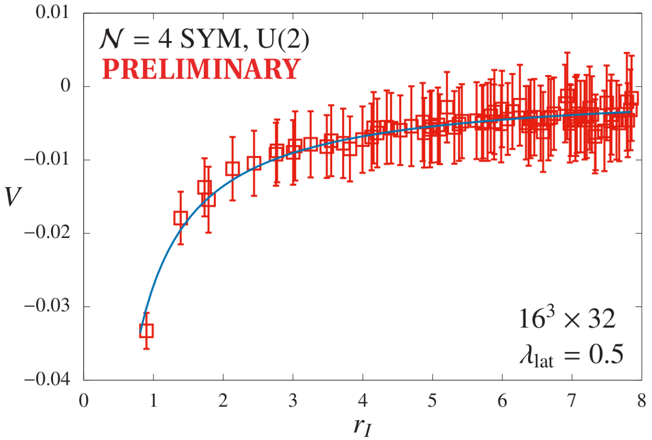
$\rightsquigarrow$



Static potential is Coulombic at all λ

Fits to confining $V(r) = A - C/r + \sigma r \longrightarrow$ vanishing string tension σ

$\Longrightarrow$ Fit to just $V(r) = A - C/r$ to extract Coulomb coefficient $C(\lambda)$

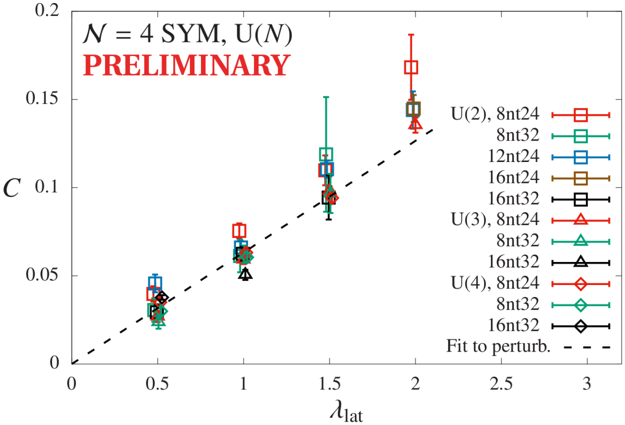


Discretization artifacts reduced by tree-level improved analysis

Coupling dependence of Coulomb coefficient

Continuum perturbation theory $\longrightarrow$ $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

Holography $\longrightarrow$ $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$ and $\lambda \to \infty$ with $\lambda \ll N$



For $\lambda_{\text{lat}} \leq 2$, consistent with leading-order perturbation theory

Konishi operator scaling dimension

$\mathcal{O}_K(x) = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}(x)\Phi^{\mathrm{I}}(x)\right]$ is simplest conformal primary operator

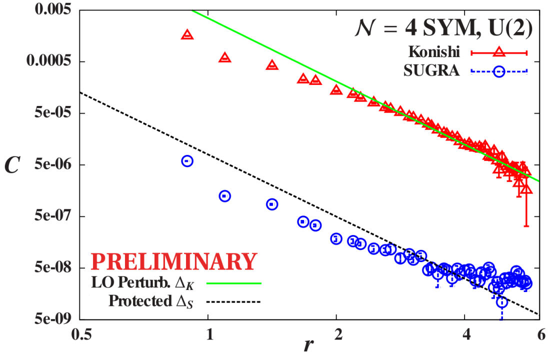
Scaling dimension $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$ investigated through perturbation theory (& S duality), holography, conformal bootstrap

$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$

'SUGRA' is 20′ op., $\Delta_S = 2$

Can compare:

- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG



Konishi operator scaling dimension

Lattice scalars $\varphi(n)$ from polar decomposition $\mathcal{U}_a(n) = e^{\varphi_a(n)} U_a(n)$

$$\mathcal{O}_K^{\text{lat}}(n) = \sum_a \text{Tr}\left[\varphi_a(n)\varphi_a(n)\right] - \text{vev} \qquad \mathcal{O}_S^{\text{lat}}(n) \sim \text{Tr}\left[\varphi_a(n)\varphi_b(n)\right]$$

$$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

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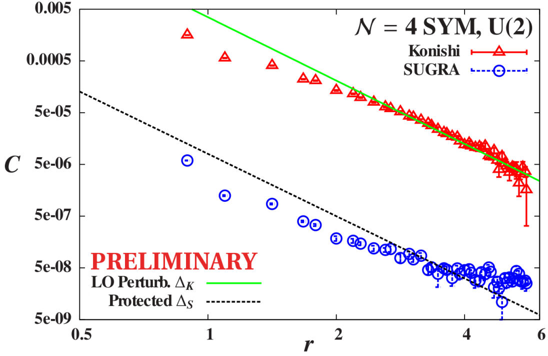
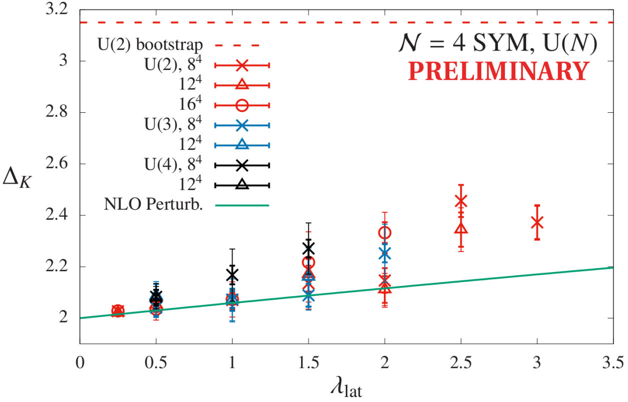


Preliminary Δ_K results from Monte Carlo RG

Analyzing both $\mathcal{O}_K^{\text{lat}}$ and $\mathcal{O}_S^{\text{lat}}$

Imposing protected $\Delta_S = 2$
$\longrightarrow \Delta_K(\lambda)$ looks perturbative

Systematic uncertainties from
different amounts of smearing



Complication from twisting $SO(4)_R \subset SO(6)_R$

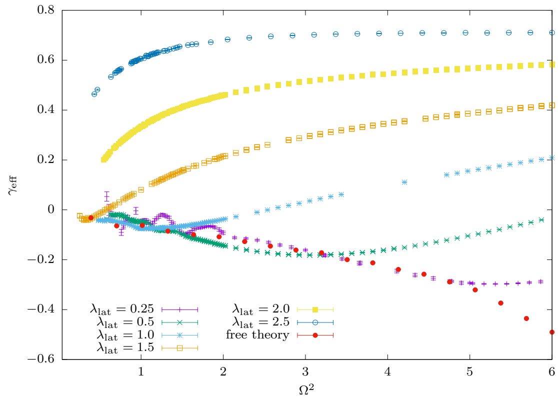
$\mathcal{O}_K^{\text{lat}}$ mixes with $SO(4)_R$-singlet part of $SO(6)_R$-nonsinglet $\mathcal{O}_S$
$\longrightarrow$ disentangle via variational analyses

Fermion bilinear anomalous dimension

Bergner–DS, to appear

Fermion op. eigenvalues predict 'mass' anomalous dimension of fermion bilinear

Should vanish $\longrightarrow$ test discretization and finite-volume effects in lattice calcs



Scale-dependent 'effective anom. dim.' due to broken conformality

Recover true critical exponent at low energy scale $\Omega^2 \ll 1$

$0.25 \leq \lambda_{\text{lat}} \leq 2.5$, with even free theory sensitive to lattice effects

Checkpoint

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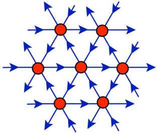
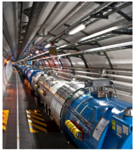
in lattice studies of supersymmetric QFTs





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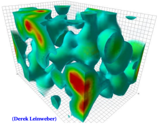
✓ Special cases: $\mathcal{N} = 1$ super-Yang–Mills; Matrix models

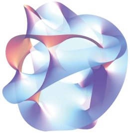


Lattice $\mathcal{N} = 4$ super-Yang–Mills

✓ Formulation highlights

✓ Dimensionally reduced (2d & 3d) thermodynamics

✓ 4d static potential & scaling dimensions — Questions?



Prospects and future directions

Future frontier: Supersymmetric QCD

Add ‘quarks’ and squarks $\longrightarrow$ investigate electric–magnetic dualities, dynamical supersymmetry breaking and more

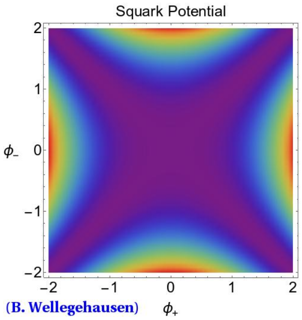
Fine-tuning back with a vengeance

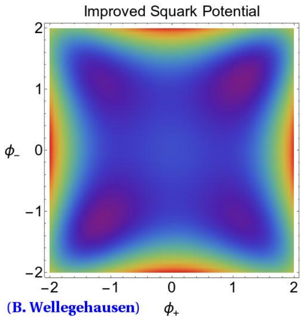
$\mathcal{O}(10)$ parameters, even using domain wall or overlap fermions [arXiv:0903.2443]

Pursuing superQCD with full fine-tuning

First step: Lattice perturbation theory as guide for future fine-tuning

Wellegehausen–Wipf, arXiv:1811.01784; Costa–Panagopoulos, arXiv:1812.06770





Alternately include only fundamental + adjoint fermions, leave scalars for future

Bergner–Piemonte, arXiv:2008.02855

Simplify superQCD: Twisted theories in 2d or 3d

Quiver construction preserves susy sub-algebra [arXiv:1505.00467]

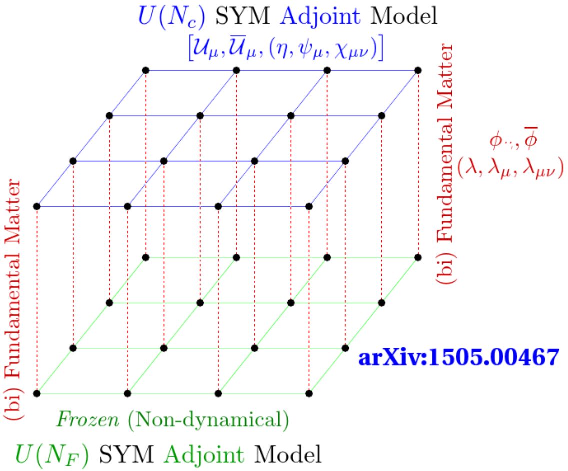
2-slice lattice SYM
with U(N) $\times$ U(F) gauge group

Adj. fields on each slice

Bi-fundamental in between

Decouple U(F) slice

$\longrightarrow$ U(N) SQCD in $d-1$ dims.
with F fund. hypermultiplets

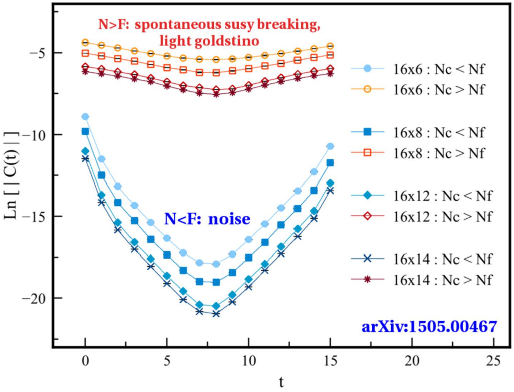


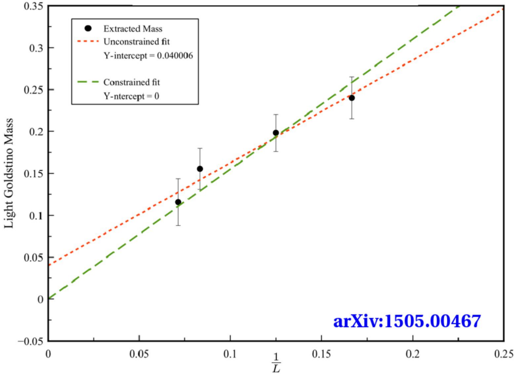
Dynamical susy breaking in 2d lattice superQCD

U(N) superQCD with F fundamental hypermultiplets

Observe spontaneous susy breaking only for $N > F$, as expected

Catterall–Veernala, arXiv:1505.00467





Future frontier: Sign problems

Recall typical algorithms sample field configurations Φ with probability $\frac{1}{\mathcal{Z}} e^{-S[\Phi]}$

$\longrightarrow$ "sign problem" if action $S[\Phi]$ can be negative or complex

Example: Spontaneous susy breaking needs vanishing Witten index

Witten index is just $\mathcal{Z} = \int \mathcal{D}\Phi \;\; e^{-S[\Phi]} \longrightarrow$ severe sign problem to have $\mathcal{Z} = 0$

Alternative approaches being explored — Joseph–Kumar, arXiv:2011.08107

Complex Langevin to be discussed by Jun Nishimura tomorrow

Quantum simulation to be discussed by Shailesh Chandrasekharan today

Future frontier: Sign problems

Recall typical algorithms sample field configurations Φ with probability $\frac{1}{\mathcal{Z}} e^{-S[\Phi]}$

$\longrightarrow$ "sign problem" if action $S[\Phi]$ can be negative or complex

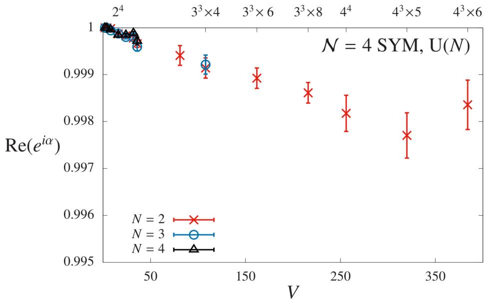
Example: $\mathcal{N} = 4$ SYM has complex pfaffian $\text{pf}\,\mathcal{D} = |\text{pf}\,\mathcal{D}| e^{i\alpha}$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}] \;\; \mathcal{O} \;\; e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]} \;\; \text{pf}\,\mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

We phase quench $\text{pf}\,\mathcal{D} \longrightarrow |\text{pf}\,\mathcal{D}|$, need to reweight $\langle \mathcal{O} \rangle = \dfrac{\left\langle \mathcal{O} e^{i\alpha} \right\rangle_{\text{pq}}}{\left\langle e^{i\alpha} \right\rangle_{\text{pq}}}$

$\Longrightarrow \left\langle e^{i\alpha} \right\rangle_{\text{pq}} = \dfrac{\mathcal{Z}}{\mathcal{Z}_{\text{pq}}}$ quantifies severity of sign problem

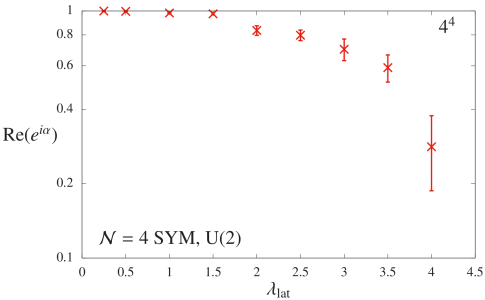
$\mathcal{N} = 4$ SYM sign problem



Fix $\lambda_{\text{lat}} = g_{\text{lat}}^2 N = 0.5$

Pfaffian nearly real positive

for all accessible volumes



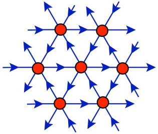
Fix 4^4 volume

Fluctuations increase with coupling

Signal-to-noise

becomes obstruction for $\lambda_{\text{lat}} \gtrsim 4$

Recap: An exciting time for lattice supersymmetry



> Significant progress currently being made
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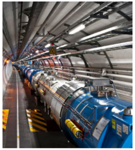
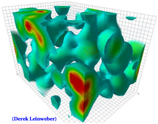
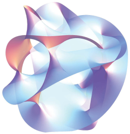


$\mathcal{N} = 1$ super-Yang–Mills avoids fine tuning from scalars



Reducing dimensions simplifies tests of holography

Preserving susy sub-algebra enables lattice $\mathcal{N} = 4$ SYM
$\longrightarrow$ thermodynamics, static potential, scaling dimensions



SuperQCD, sign problems and much more to do in the future

Thanks for your attention! Any further questions?

Collaborators

Georg Bergner, Simon Catterall, Joel Giedt, Raghav Jha, Anosh Joseph, Angel Sherletov, Toby Wiseman

Funding and computing resources

UK Research and Innovation

Backup: Breakdown of Leibniz rule on the lattice

$\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$ is problematic

$\Longrightarrow$ try finite difference $\partial\phi(x) \longrightarrow \Delta\phi(x) = \frac{1}{a}\left[\phi(x+a) - \phi(x)\right]$

Crucial difference between ∂ and Δ

$$\begin{aligned}\Delta\left[\phi\eta\right] &= a^{-1}\left[\phi(x+a)\eta(x+a) - \phi(x)\eta(x)\right] \\ &= \left[\Delta\phi\right]\eta + \phi\Delta\eta + a\left[\Delta\phi\right]\Delta\eta\end{aligned}$$

Full supersymmetry requires Leibniz rule $\partial\left[\phi\eta\right] = \left[\partial\phi\right]\eta + \phi\partial\eta$

only recoverd in $a \to 0$ continuum limit for any local finite difference

Backup: Breakdown of Leibniz rule on the lattice

Full supersymmetry requires Leibniz rule $\partial\left[\phi\eta\right] = \left[\partial\phi\right]\eta + \phi\partial\eta$

only recoverd in $a \to 0$ continuum limit for any local finite difference

Supersymmetry vs. locality 'no-go' theorems

by Kato–Sakamoto–So [arXiv:0803.3121] and Bergner [arXiv:0909.4791]

Complicated constructions to balance locality vs. supersymmetry

Non-ultralocal product operator $\longrightarrow$ lattice Leibniz rule but not gauge invariance

D'Adda–Kawamoto–Saito, arXiv:1706.02615

Cyclic Leibniz rule $\longrightarrow$ partial lattice supersymmetry but only (0+1)d QM so far

Kadoh–Kamei–So, arXiv:1904.09275

Backup: Complexified gauge field from twisting

Combining A_μ and $\Phi^{\mathrm{I}} \longrightarrow \mathcal{A}_a$ and $\overline{\mathcal{A}}_a$

produces $\mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1)$ gauge theory

Complicates lattice action but needed so that $\mathcal{Q}\, \mathcal{A}_a = \psi_a$

Further motivation: Under $\mathrm{SO}(d)_{\mathrm{tw}} = \mathrm{diag}\left[\mathrm{SO}(d)_{\mathrm{euc}} \otimes \mathrm{SO}(d)_R\right]$

$$
\begin{aligned}
A_\mu &\sim \text{vector} \otimes \text{scalar} = \text{vector} \\
\Phi^{\mathrm{I}} &\sim \text{scalar} \otimes \text{vector} = \text{vector}
\end{aligned}
$$

Easiest to see in 5d (then dimensionally reduce)

$$
\mathcal{A}_a = A_a + i\Phi_a \longrightarrow (A_\mu, \phi) + i(\Phi_\mu, \overline{\phi})
$$

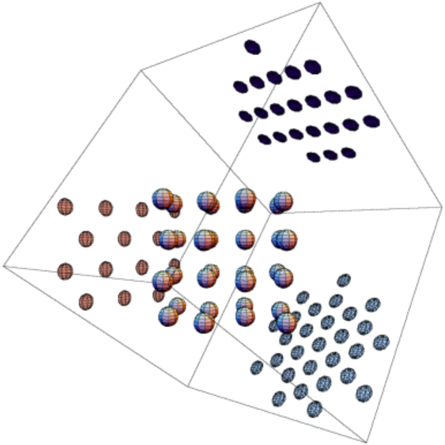
Backup: A_4^* lattice from five dimensions

Again dimensionally reduce, treating all five gauge links symmetrically

Start with hypercubic lattice
in 5d momentum space

Symmetric constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in position space



Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

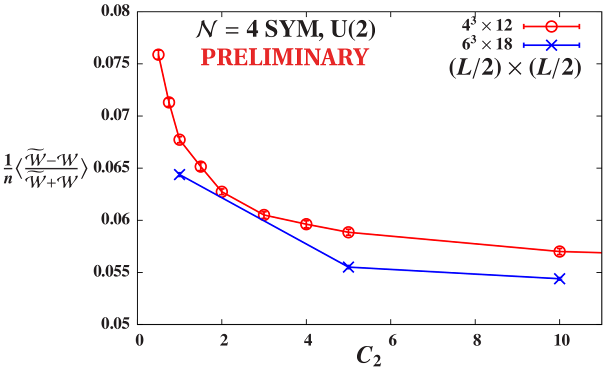
$$\text{“}\mathcal{Q} \; + \; \text{discrete} \;\; R_a \subset \mathrm{SO}(4)_{\mathrm{tw}} \; = \; \mathcal{Q}_a \;\; \text{and} \;\; \mathcal{Q}_{ab}\text{”}$$

[arXiv:1306.3891]

Test R_a on Wilson loops

$$\widetilde{\mathcal{W}}_{ab} \equiv R_a \mathcal{W}_{ab}$$

Tune coeff. c_2 of d^2 term in action
for fastest restoration
towards continuum limit



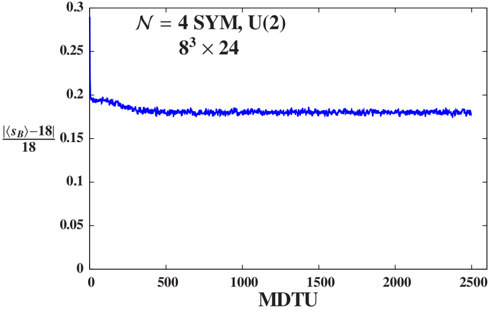
Backup: Problem with SU(N) flat directions

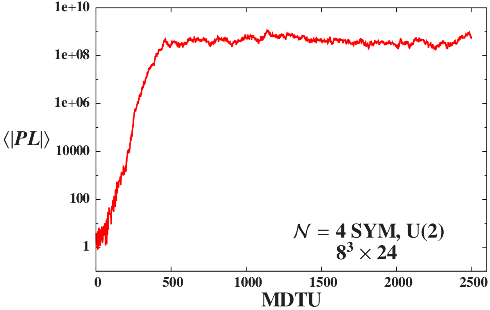
$\mu^2/\lambda_{\text{lat}}$ too small $\longrightarrow$ $\mathcal{U}_a$ can move far from continuum form $\mathbb{I}_N + \mathcal{A}_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 2.5$ on $8^3 \times 24$ volume

Left: Bosonic action stable $\sim$18% off its supersymmetric value

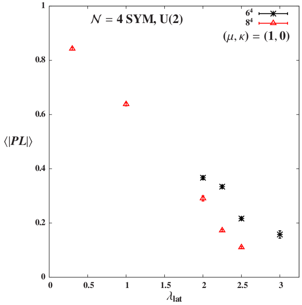
Right: (Complexified) Polyakov loop wanders off to $\sim 10^9$

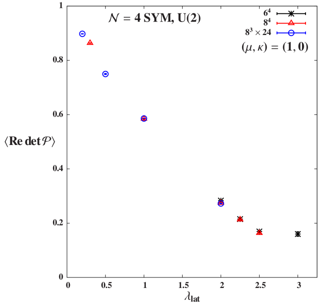


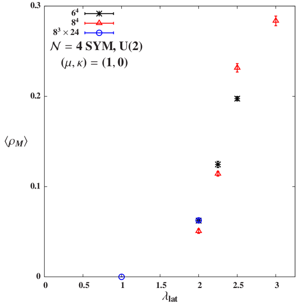


Backup: Problem with U(1) flat directions

Monopole condensation $\longrightarrow$ confined lattice phase not present in continuum







Around the same $2\lambda_{\text{lat}} \approx 2\ldots$

Left: Polyakov loop falls towards zero

Center: Plaquette determinant falls towards zero

Right: Density of U(1) monopole world lines becomes non-zero

Backup: Regulating SU(N) flat directions

Add soft $\mathcal{Q}$-breaking scalar potential to lattice action

$$S = \frac{N}{4\lambda_{\text{lat}}}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V\right]$$

$$V = \sum_a\left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2$$ lifts SU(N) flat directions,

ensures $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

Correct continuum limit requires $\mu^2 \to 0$ to restore $\mathcal{Q}$ and recover moduli space

Typically scale $\mu \propto 1/L$ in $L \to \infty$ continuum extrapolation

Backup: Naively regulating U(1) flat directions

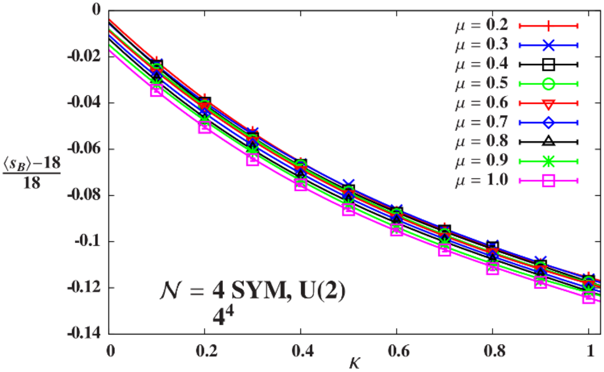
In earlier work we added another soft $\mathcal{Q}$-breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}}\mu^2 \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2 + \kappa \sum_{a<b} \left|\det \mathcal{P}_{ab} - 1\right|^2$$

More sensitivity to κ than to μ^2

Showing $\mathcal{Q}$ Ward identity
from bosonic action

$$\langle s_B \rangle = 9N^2/2$$

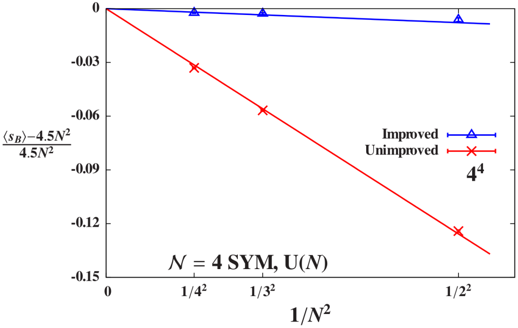


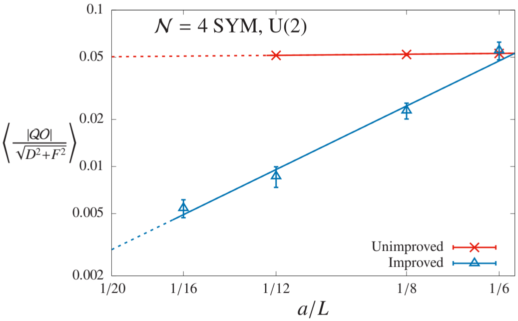
Backup: Better regulating U(1) flat directions

$$S = \frac{N}{4\lambda_{\text{lat}}}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\left\{\overline{\mathcal{D}}_a\mathcal{U}_a + G\sum_{a<b}\left[\det\mathcal{P}_{ab} - 1\right]\mathbb{I}_N\right\} - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V\right]$$

$\mathcal{Q}$ Ward identity violations scale $\propto 1/N^2$ (**left**) and $\propto (a/L)^2$ (**right**)

$\sim$ effective '$\mathcal{O}(a)$ improvement' since $\mathcal{Q}$ forbids all dim-5 operators





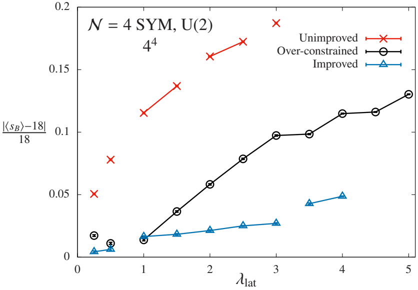
Backup: Supersymmetric moduli space modification [arXiv:1505.03135]

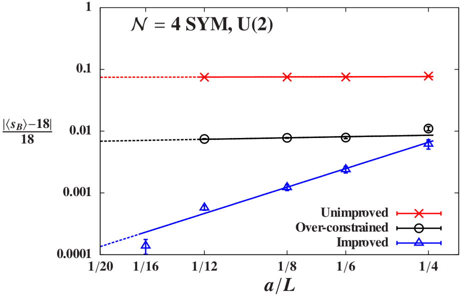
Method to impose $\mathcal{Q}$-invariant constraints on generic site operator $\mathcal{O}(n)$

Modify auxiliary field equations of motion $\longrightarrow$ moduli space

$$d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) \qquad \longrightarrow \qquad d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

However, both U(1) and SU(N) $\in \mathcal{O}(n)$ over-constrains system





Backup: Dimensional reduction to 2d $\mathcal{N} = (8, 8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

$A_4^* \longrightarrow A_2^*$ (triangular) lattice

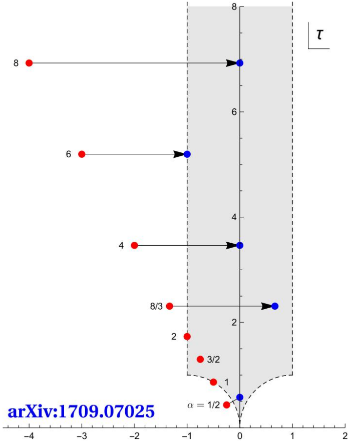
Torus **skewed** depending on $\alpha = L/N_t$

Modular transformation into fundamental domain

$\longrightarrow$ some skewed tori actually rectangular

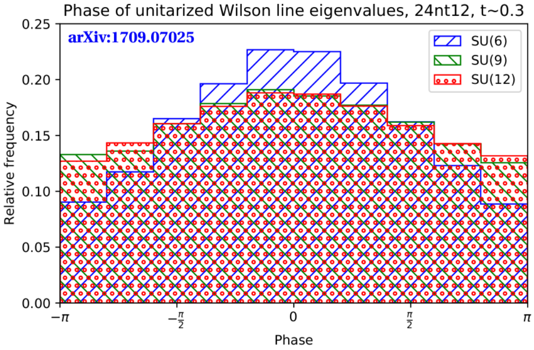
Also need to stabilize compactified links

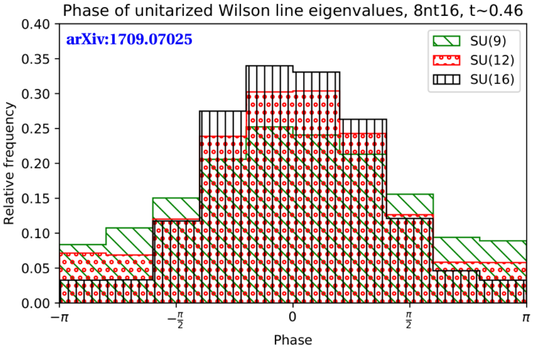
to ensure broken center symmetries



Backup: 2d $\mathcal{N} = (8,8)$ SYM Wilson line eigenvalues

Check ‘spatial deconfinement’ through Wilson line eigenvalue phases



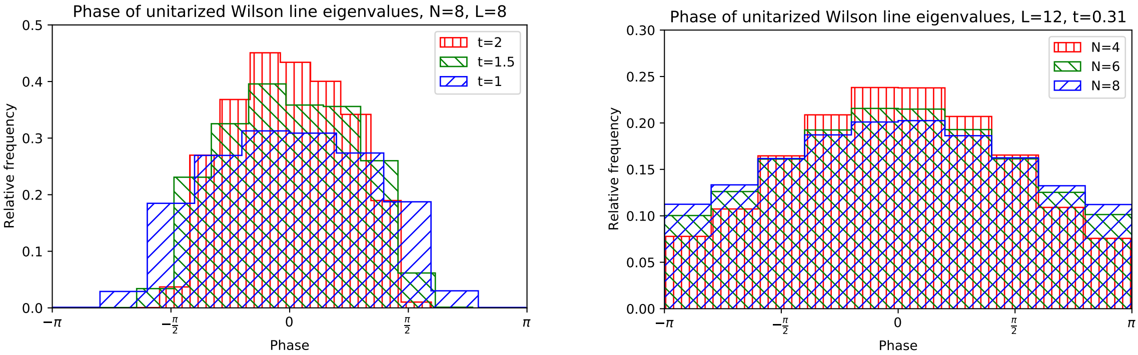


Left: $\alpha = 2$ distributions more extended as *N* increases $\longrightarrow$ D1 black string

Right: $\alpha = 1/2$ distributions more compact as *N* increases $\longrightarrow$ D0 black hole

Backup: 3d $\mathcal{N} = 8$ SYM Wilson line eigenvalues

Check ‘spatial deconfinement’ through Wilson line eigenvalue phases

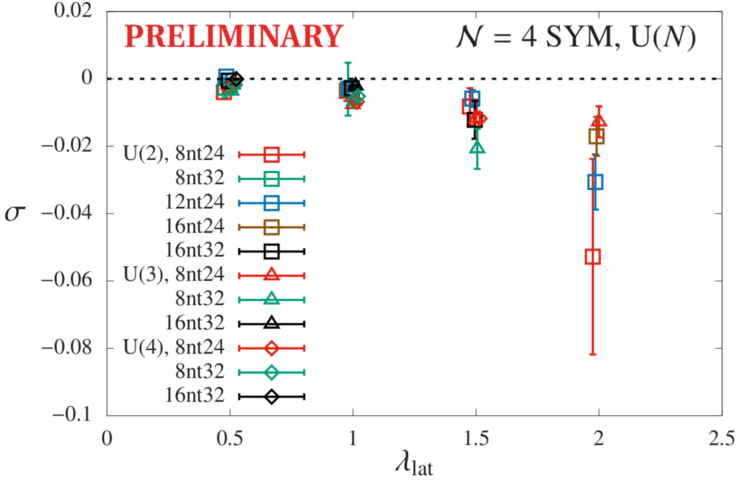


Left: High-temperature U(8) 8^3 distributions more compact as t increases

Right: Low-temperature U(N) 12^3 distributions more uniform as N increases

Backup: Static potential is Coulombic at all λ

String tension σ from fits to confining form $V(r) = A - C/r + \sigma r$

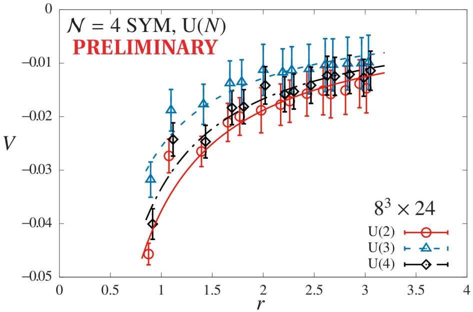


Slightly negative values flatten $V(r_I)$ for $r_I \lesssim L/2$

$\sigma \to 0$ as accessible range of r_I increases on larger volumes

Backup: Discretization artifacts in static potential

Discretization artifacts visible at short distances
where Coulomb term in $V(r) = A - C/r$ is most significant



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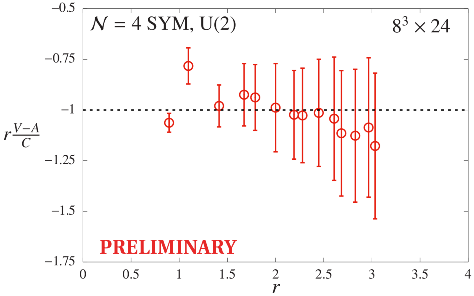


Danger of distorting Coulomb coefficient C

Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential

Associate $V(r_\nu)$ data with 'r_I' from Fourier transform of gluon propagator

Recall $\displaystyle \frac{1}{4\pi^2 r^2} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{e^{ir_\nu k_\nu}}{k^2}$ where $\displaystyle \frac{1}{k^2} = G(k_\nu)$ in continuum

$$A_4^* \text{ lattice} \longrightarrow \frac{1}{r_I^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4\widehat{k}}{(2\pi)^4} \frac{\cos\left(ir_\nu \widehat{k}_\nu\right)}{4\sum_{\mu=1}^{4} \sin^2\left(\widehat{k}\cdot\widehat{e}_\mu \;/\; 2\right)}$$

Tree-level lattice propagator from arXiv:1102.1725

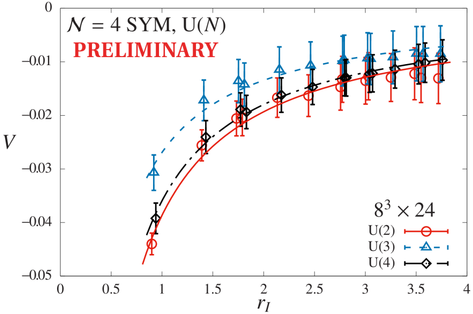
$\widehat{e}_\mu$ are A_4^* lattice basis vectors;

momenta $\widehat{k} = \frac{2\pi}{L}\sum_{\mu=1}^{4} n_\mu \widehat{g}_\mu$ depend on dual basis vectors

Backup: Tree-level-improved static potential

$$\frac{1}{r_I^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4\widehat{k}}{(2\pi)^4} \frac{\cos\left(ir_\nu \widehat{k}_\nu\right)}{4\sum_{\mu=1}^{4} \sin^2\left(\widehat{k}\cdot\widehat{e}_\mu \;/\; 2\right)}$$

$\longrightarrow$ significantly reduced discretization artifacts



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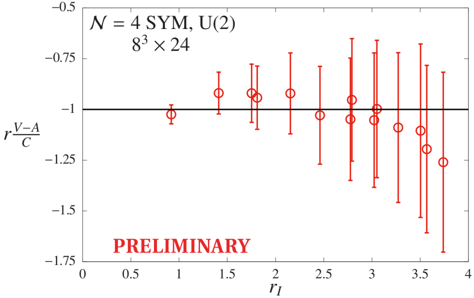


Backup: Scaling dimensions from MCRG stability matrix

Lattice system: $H = \sum_i c_i\, \mathcal{O}_i$ (infinite sum)

Couplings flow under RG blocking $\longrightarrow$ $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Conformal fixed point $\longrightarrow$ $H^\star = R_b H^\star$ with couplings $c_i^\star$

Linear expansion around fixed point $\longrightarrow$ stability matrix $T_{ik}^\star$

$$c_i^{(n)} - c_i^\star = \sum_k \left.\frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}}\right|_{H^\star} \left(c_k^{(n-1)} - c_k^\star\right) \equiv \sum_k T_{ik}^\star \left(c_k^{(n-1)} - c_k^\star\right)$$

Correlators of $\mathcal{O}_i$, $\mathcal{O}_k$ $\longrightarrow$ elements of stability matrix [Swendsen, 1979]

Eigenvalues of $T_{ik}^\star$ $\longrightarrow$ scaling dimensions of corresponding operators

Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $\mathcal{Q}$ and S_5 symmetries $\longleftrightarrow$ geometric structure

Simple transformation constructed in arXiv:1408.7067

$$\mathcal{U}'_a(n') = \xi\, \mathcal{U}_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) \qquad\qquad \eta'(n') = \eta(n)$$

$$\psi'_a(n') = \xi\left[\psi_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) + \mathcal{U}_a(n)\psi_a(n+\widehat{\mu}_a)\right] \qquad\qquad \text{etc.}$$

Doubles lattice spacing $a \longrightarrow a' = 2a$, with tunable rescaling factor ξ

Scalar fields from polar decomposition $\mathcal{U}(n) = e^{\varphi(n)} U(n)$

$\Longrightarrow$ shift $\varphi \longrightarrow \varphi + \log\xi$ to keep blocked U unitary

$\mathcal{Q}$-preserving RG transformation needed

to show only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Backup: Smearing for Konishi analyses

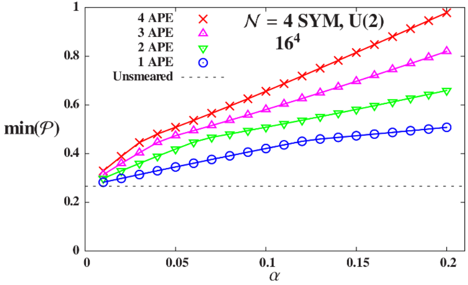
Smear to enlarge (MCRG or variational) operator basis

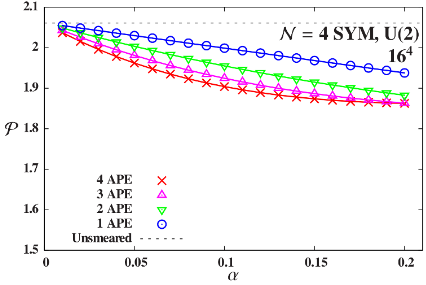
APE-like smearing: $\quad — \quad \longrightarrow \quad (1-\alpha)— \quad + \quad \frac{\alpha}{8}\sum \sqcap,$

staples built from unitary parts of links but no final unitarization

Average plaquette stable upon smearing (**right**),

minimum plaquette steadily increases (**left**)





Backup: More on dynamical susy breaking

Spontaneous susy breaking means $\langle 0\,|H|\,0\rangle > 0$ or equivalently $\langle \mathcal{Q}\mathcal{O}\rangle \neq 0$

Twisted superQCD auxiliary field e.o.m. $\longleftrightarrow$ Fayet–Iliopoulos D-term potential

$$d = \overline{\mathcal{D}}_a\mathcal{U}_a + \sum_{i=1}^{F} \phi_i\overline{\phi}_i - r\mathbb{I}_N \qquad \longleftrightarrow \qquad \text{Tr}\left[\left(\sum\nolimits_i \phi_i\overline{\phi}_i - r\mathbb{I}_N\right)^2\right] \in H$$

Have $F \times N$ scalar vevs to zero out $N \times N$ matrix

$\longrightarrow N > F$ suggests susy breaking, $\langle 0\,|H|\,0\rangle > 0 \longleftrightarrow \langle \mathcal{Q}\eta\rangle = \langle d\rangle \neq 0$