Maximally supersymmetric Yang–Mills on the lattice

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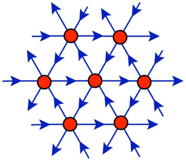
Southampton String Theory Seminar, 27 November 2019

arXiv:1611.06561 arXiv:1709.07025 arXiv:1810.09282

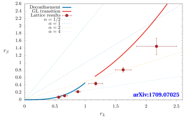
and more to come with Simon Catterall, Raghav Jha and Toby Wiseman

Overview and plan



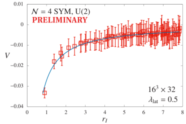
Why: Lattice supersymmetry

How: Lattice formulation highlights



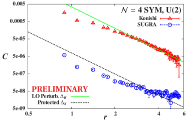
What: Recent results

Dimensionally reduced (2d) thermodynamics



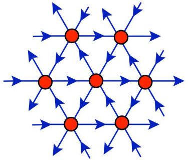
Static potential (4d)

Conformal scaling dimensions



Prospects and future directions

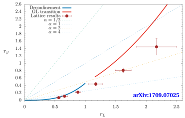
Overview and plan

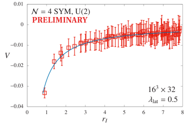


Central idea

Preserve (some) susy in discrete space-time

$\longrightarrow$ practical lattice investigations



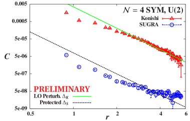


Goals

1) Reproduce reliable results

in perturbative and holographic regimes

2) Access new domains

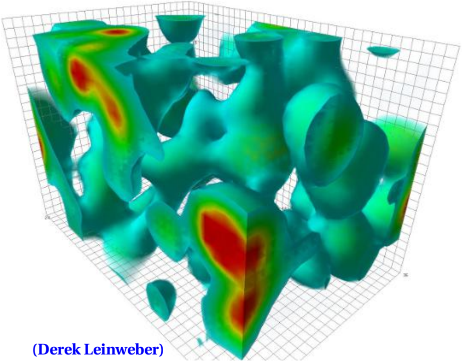


Motivations

Lattice field theory promises first-principles predictions
for strongly coupled supersymmetric QFTs

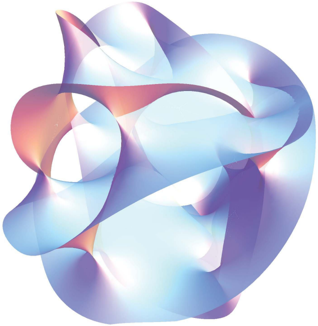
BSM

QFT



(Derek Leinweber)

Holography



Supersymmetry must be broken on the lattice

Supersymmetry is a space-time symmetry, $(\mathrm{I} = 1, \cdots, \mathcal{N})$

adding spinor generators Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ to translations, rotations, boosts

$\left\{Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{J}}_{\dot{\alpha}}\right\} = 2\delta^{\mathrm{IJ}}\sigma^{\mu}_{\alpha\dot{\alpha}}P_{\mu}$ broken in discrete space-time

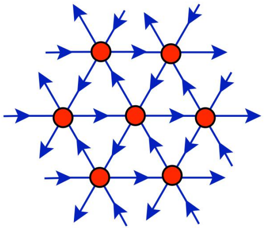
$\longrightarrow$ relevant susy-violating operators

Supersymmetry need not be *completely* broken on the lattice

Preserve susy sub-algebra at non-zero lattice spacing

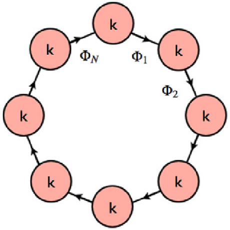
$\Longrightarrow$ correct continuum limit with little or no fine tuning

Equivalent constructions from 'topological' twisting and dim'l deconstruction



Review:

arXiv:0903.4881



Need 2^d supersymmetries in d dimensions

$d = 4 \longrightarrow$ maximally supersymmetric Yang–Mills ($\mathcal{N} = 4$ SYM)

$\mathcal{N} = 4$ SYM in a nutshell

Arguably simplest non-trivial 4d QFT $\longrightarrow$ dualities, amplitudes, ...

SU(N) gauge theory with $\mathcal{N} = 4$ fermions Ψ^{I} and 6 scalars Φ^{IJ},
all massless and in adjoint rep.

Symmetries relate coefficients of kinetic, Yukawa and Φ^4 terms

Maximal 16 supersymmetries Q_α^{I} and $\overline{Q}_{\dot{\alpha}}^{\mathrm{I}}$ $\quad \mathrm{I} = 1, \cdots, 4$
transform under global SU(4) $\sim$ SO(6) R symmetry

Conformal $\longrightarrow$ β function is zero for all values of $\lambda = g^2 N$

Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}} \gamma_5$$
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$
with $a, b = 1, \cdots, 5$

R-symmetry index $\times$ Lorentz index $\Longrightarrow$ reps of 'twisted rotation group'

$$\text{SO}(4)_{tw} \equiv \text{diag}\left[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R\right] \qquad \text{SO}(4)_R \subset \text{SO}(6)_R$$

Change of variables $\longrightarrow$ $\mathcal{Q}$ transform with integer 'spin' under $\text{SO}(4)_{tw}$

Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}} \gamma_5$$

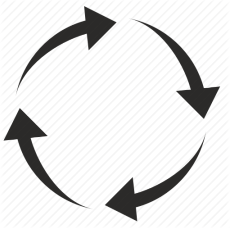
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$

with $a, b = 1, \cdots, 5$

Discrete space-time

Can preserve closed sub-algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$



Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q_\alpha^1 & Q_\alpha^2 & Q_\alpha^3 & Q_\alpha^4 \\ \\ \overline{Q}_{\dot\alpha}^1 & \overline{Q}_{\dot\alpha}^2 & \overline{Q}_{\dot\alpha}^3 & \overline{Q}_{\dot\alpha}^4 \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}} \gamma_5$$

$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$

with $a, b = 1, \cdots, 5$

Discrete space-time

Can preserve closed sub-algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$

Completing the twist

Fields also transform with integer spin under $\text{SO}(4)_{tw}$ — no spinors

$$\Psi \text{ and } \overline{\Psi} \longrightarrow \eta,\ \psi_a \text{ and } \chi_{ab}$$

$$A_\mu \text{ and } \Phi^{\mathrm{I}} \longrightarrow \text{complexified gauge field } \mathcal{A}_a \text{ and } \overline{\mathcal{A}}_a$$

$$\longrightarrow \text{U}(N) = \text{SU}(N) \otimes \text{U}(1) \text{ gauge theory}$$

✓ $\mathcal{Q}$ interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\mathcal{Q}\,\mathcal{A}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

$\nwarrow$ bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Covariant derivatives $\longrightarrow$ finite difference operators

Complexified gauge fields $\mathcal{A}_a$ $\longrightarrow$ gauge links $\mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C})$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

Geometry: η on sites, ψ_a on links, etc.

Supersymmetric lattice action $(\mathcal{Q}S = 0)$ from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity

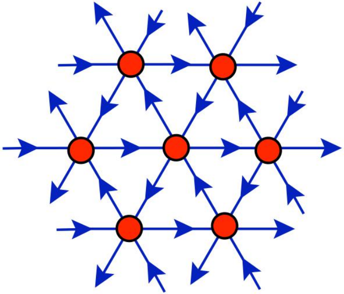
$$S = \frac{N}{4\lambda_{\text{lat}}}\text{Tr}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de}\right]$$

Five links in four dimensions $\longrightarrow$ A_4^* lattice

A_4^* $\sim$ 4d analog of 2d triangular lattice

Basis vectors linearly dependent and non-orthogonal

Large S_5 point group symmetry



S_5 irreps precisely match onto irreps of twisted $\mathrm{SO}(4)_{tw}$

$$\psi_a \longrightarrow \psi_\mu, \quad \overline{\eta} \qquad \text{is} \qquad \mathbf{5} \longrightarrow \mathbf{4} \oplus \mathbf{1}$$

$$\chi_{ab} \longrightarrow \chi_{\mu\nu}, \quad \overline{\psi}_\mu \qquad \text{is} \qquad \mathbf{10} \longrightarrow \mathbf{6} \oplus \mathbf{4}$$

$S_5 \longrightarrow \mathrm{SO}(4)_{tw}$ in continuum limit restores $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Checkpoint

Analytic results for twisted $\mathcal{N} = 4$ SYM on A_4^* lattice

U(N) gauge invariance + $\mathcal{Q}$ + S_5 lattice symmetries

$\longrightarrow$ Moduli space preserved to all orders

$\longrightarrow$ One-loop lattice β function vanishes

$\longrightarrow$ Only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

[arXiv:1102.1725, arXiv:1306.3891, arXiv:1408.7067]

Not yet suitable for numerical calculations

Must regulate zero modes and flat directions, especially in U(1) sector

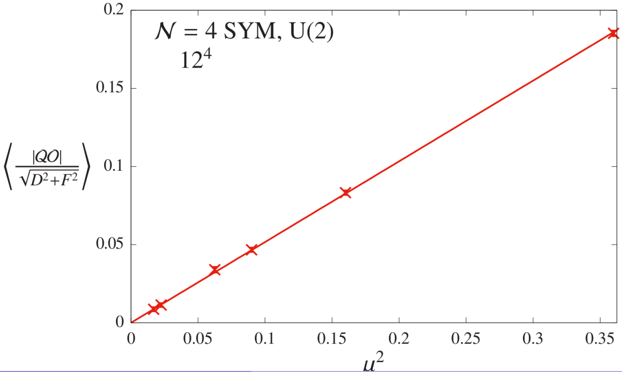
Two deformations stabilize lattice calculations

(i) Add SU(N) scalar potential $\propto \mu^2 \sum_a \left(\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - N\right)^2$

Softly breaks susy $\longrightarrow$ $\mathcal{Q}$-violating operators vanish $\propto \mu^2 \to 0$

Test via Ward identity violations

$$\mathcal{Q}\left[\eta\mathcal{U}_a\overline{\mathcal{U}}_a\right] \neq 0$$



Two deformations stabilize lattice calculations

(ii) Constrain U(1) plaquette determinant $\sim G \sum_{a<b} \left(\det \mathcal{P}_{ab} - 1\right)$

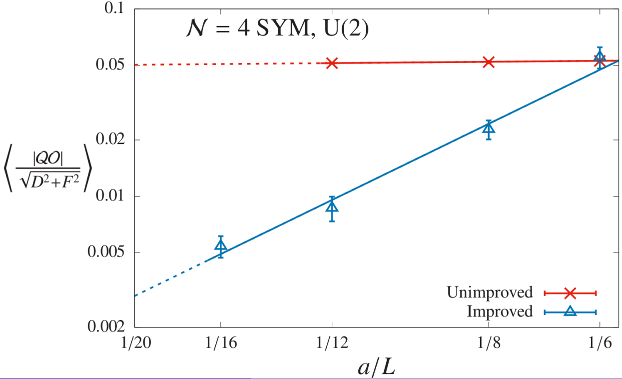
Implemented supersymmetrically as Fayet–Iliopoulos D-term potential

Test via Ward identity violations

$$\mathcal{Q}\left[\eta \mathcal{U}_a \overline{\mathcal{U}}_a\right] \neq 0$$

Log–log axes

$\longrightarrow$ violations $\propto (a/L)^2$



Advertisement: Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$S_{\text{imp}} = S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \tag{18}$$

$$S'_{\text{exact}} = \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr}\left[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_a\psi_a(n) + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a(n) + G\sum_{a\neq b}\left(\det\mathcal{P}_{ab}(n) - 1\right)\mathbb{I}_N\right)^2\right] - S_{\text{det}}$$

$$S_{\text{det}} = \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr}\left[\eta(n)\right] \sum_{a\neq b}\left[\det\mathcal{P}_{ab}(n)\right] \text{Tr}\left[\mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n+\widehat{\mu}_b)\psi_a(n+\widehat{\mu}_b)\right]$$

$$S_{\text{closed}} = -\frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr}\left[\epsilon_{abcde}\ \chi_{de}(n+\widehat{\mu}_a+\widehat{\mu}_b+\widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_c\chi_{ab}(n)\right],$$

$$S'_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}}\mu^2 \sum_n \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right)^2$$

$\gtrsim$100 inter-node data transfers in the fermion operator — non-trivial. . .

Public parallel code to reduce barriers to entry: github.com/daschaich/susy

Evolved from MILC QCD code, user guide in arXiv:1410.6971

(i) Thermodynamics on $(r_L \times r_\beta)$ 2-torus

arXiv:1709.07025

Dimensionally reduce to (deconfined) 2d $\mathcal{N} = (8, 8)$ SYM with four scalar $\mathcal{Q}$

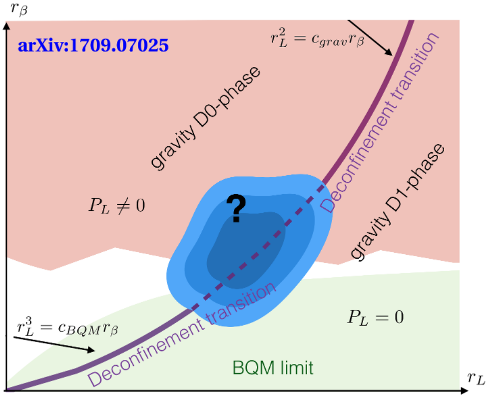
Low temperatures $t = 1/r_\beta \longleftrightarrow$ black holes in dual supergravity

For decreasing r_L **at large *N***

homogeneous black string (D1)
$\longrightarrow$ localized black hole (D0)

$\updownarrow$

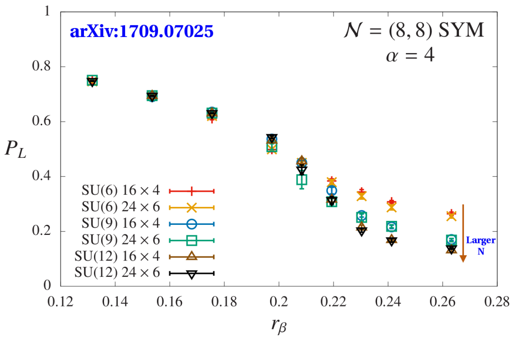
"spatial deconfinement"
signalled by Wilson line P_L

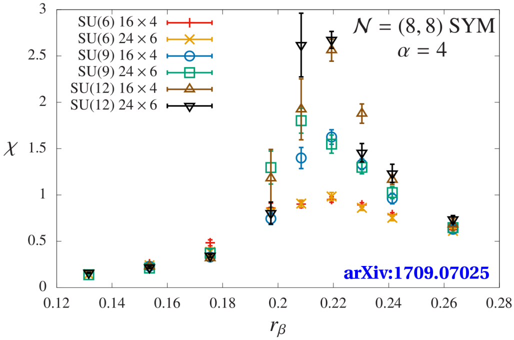


Spatial deconfinement transition signals

Peaks in Wilson line susceptibility match change in its magnitude $|\mathrm{PL}|$,
grow with size of SU(N) gauge group, comparing $N = 6, 9, 12$

Agreement for 16×4 vs. 24×6 lattices (aspect ratio $\alpha = r_L/r_\beta = 4$)

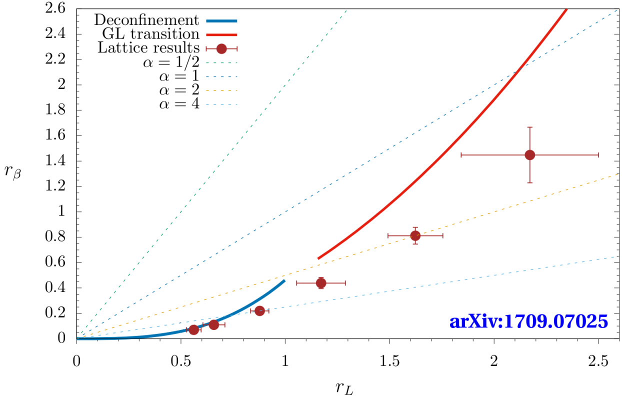




Lattice 2d $\mathcal{N} = (8,8)$ SYM phase diagram

Large $\alpha = r_L/r_\beta \gtrsim 4 \longrightarrow$ good agreement with high-temperature bosonic QM

Small $\alpha \lesssim 2 \longrightarrow$ harder to control uncertainties with $6 \le N \le 16$



Overall consistent with holography

Comparing multiple lattice sizes

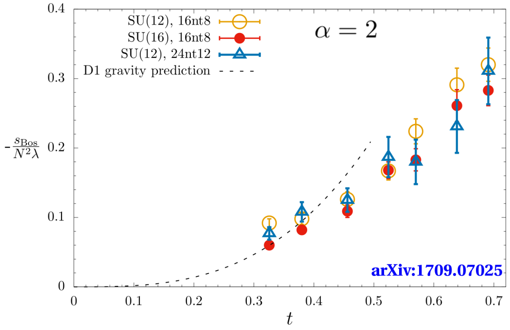
Controlled extrapolations are work in progress

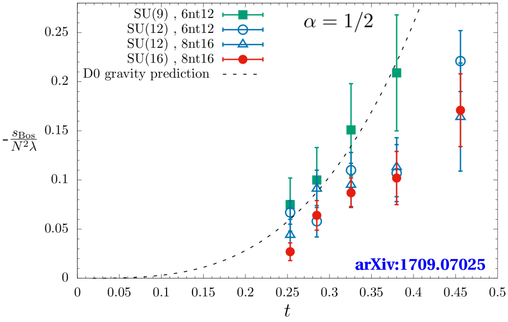
Dual black hole thermodynamics

Dual black hole energy from 2d $\mathcal{N} = (8,8)$ SYM

$\propto t^3$ for large-r_L D1 phase $\propto t^{3.2}$ for small-r_L D0 phase

Lattice results consistent with holography for sufficiently low $t \lesssim 0.4$

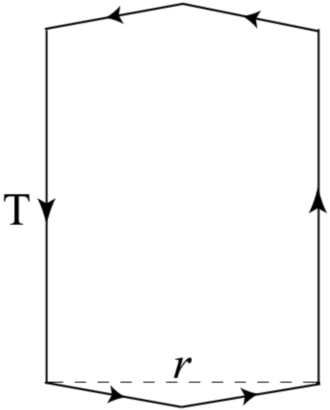




(ii) 4d $\mathcal{N} = 4$ SYM static potential $V(r)$

Static probes $\longrightarrow$ $r \times T$ Wilson loops $W(r, T) \propto e^{-V(r)\,T}$

Coulomb gauge trick reduces A_4^* lattice complications



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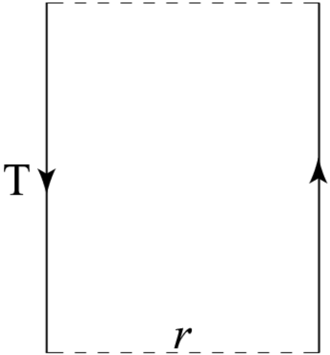
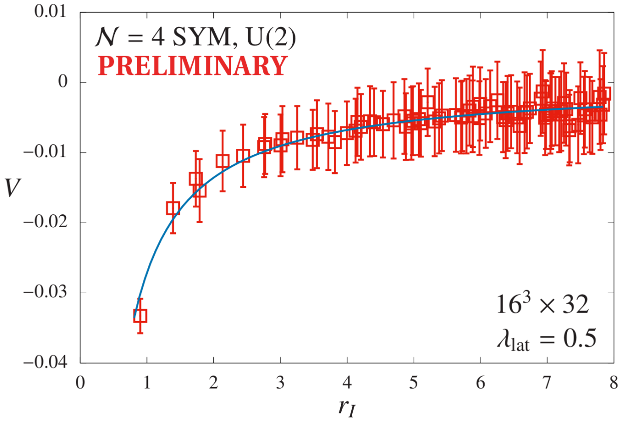


Static potential is Coulombic at all λ

Fits to confining $V(r) = A - C/r + \sigma r \longrightarrow$ vanishing string tension σ

$\Longrightarrow$ Fit to just $V(r) = A - C/r$
to extract Coulomb coefficient $C(\lambda)$

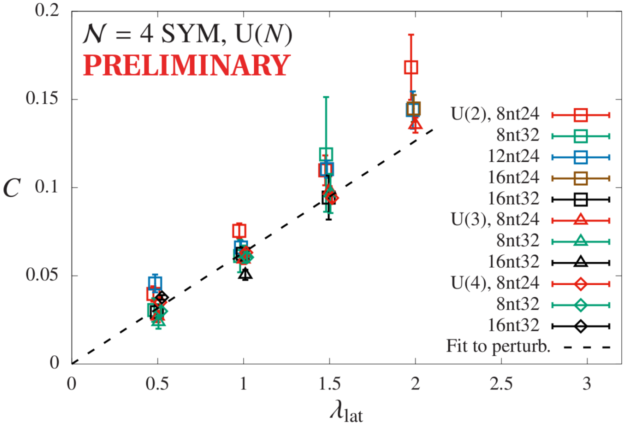
Discretization artifacts reduced
by tree-level improved analysis



Coupling dependence of Coulomb coefficient

Continuum perturbation theory $\longrightarrow$ $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

Holography $\longrightarrow$ $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$ and $\lambda \to \infty$ with $\lambda \ll N$



For $\lambda_{\text{lat}} \leq 2$, consistent with leading-order perturbation theory

(iii) Konishi operator scaling dimension

$\mathcal{O}_K(x) = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}(x)\Phi^{\mathrm{I}}(x)\right]$ is simplest conformal primary operator

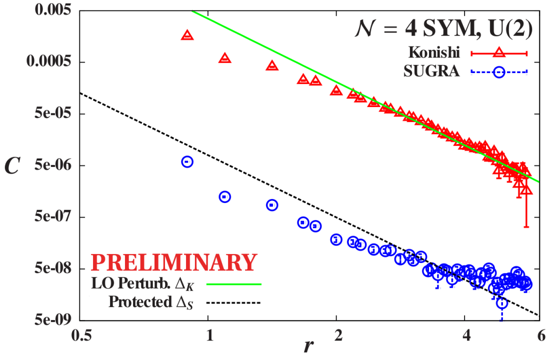
Scaling dimension $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$ investigated through perturbation theory (& S duality), holography, conformal bootstrap

$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$

'SUGRA' is $20'$ op., $\Delta_S = 2$

Will compare:

- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG



(iii) Konishi operator scaling dimension

Lattice scalars $\varphi(n)$ from polar decomposition $\mathcal{U}_a(n) \longrightarrow e^{\varphi_a(n)} U_a(n)$

$$\mathcal{O}_K^{\text{lat}}(n) = \sum_a \text{Tr}\left[\varphi_a(n)\varphi_a(n)\right] - \text{vev} \qquad\qquad \mathcal{O}_S^{\text{lat}}(n) \sim \text{Tr}\left[\varphi_a(n)\varphi_b(n)\right]$$

$$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

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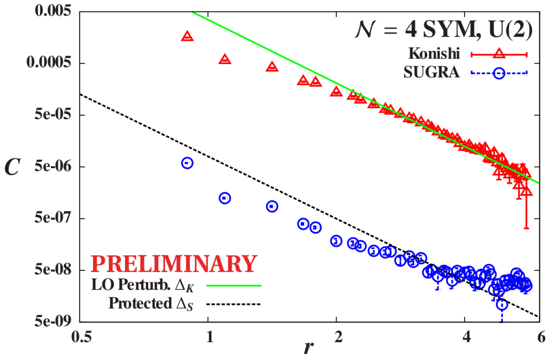


Scaling dimensions from MCRG stability matrix

Lattice system: $H = \sum_i c_i \, \mathcal{O}_i$ (infinite sum)

Couplings flow under RG blocking $\longrightarrow$ $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Fixed point $\longrightarrow$ $H^\star = R_b H^\star$ with couplings $c_i^\star$

Linear expansion around fixed point $\longrightarrow$ stability matrix $T_{ik}^\star$

$$c_i^{(n)} - c_i^\star = \sum_k \left. \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \right|_{H^\star} \left(c_k^{(n-1)} - c_k^\star\right) \equiv \sum_k T_{ik}^\star \left(c_k^{(n-1)} - c_k^\star\right)$$

Correlators of $\mathcal{O}_i$, $\mathcal{O}_k$ $\longrightarrow$ elements of stability matrix [Swendsen, 1979]

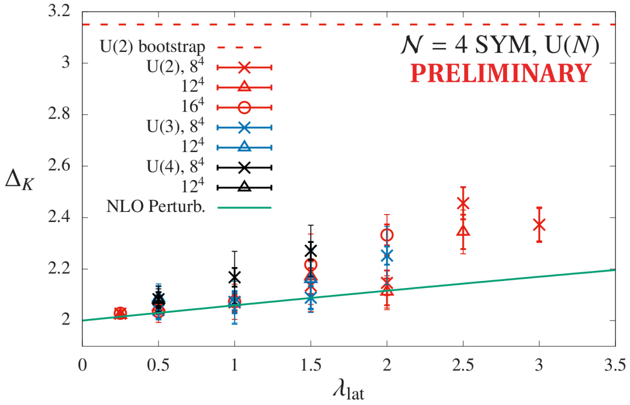
Eigenvalues of $T_{ik}^\star$ $\longrightarrow$ scaling dimensions of corresponding operators

Preliminary Δ_K results from Monte Carlo RG

Analyzing both $\mathcal{O}_K^{\text{lat}}$ and $\mathcal{O}_S^{\text{lat}}$

Imposing protected $\Delta_S = 2$
$\longrightarrow \Delta_K(\lambda)$ looks perturbative

Systematic uncertainties from
different amounts of smearing



Complication from twisting $SO(4)_R \subset SO(6)_R$

$\mathcal{O}_K^{\text{lat}}$ mixes with $SO(4)_R$-singlet part of $SO(6)_R$-nonsinglet $\mathcal{O}_S$
$\longrightarrow$ disentangle via variational analyses

Future: Pushing $\mathcal{N} = 4$ SYM to stronger coupling

✓ Reproduce reliable 4d results in perturbative regime

$\longrightarrow$ Check holographic predictions and access new domains

Sign problem seems to become obstruction

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}] \;\; \mathcal{O} \;\; e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]} \;\; \text{pf}\, \mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

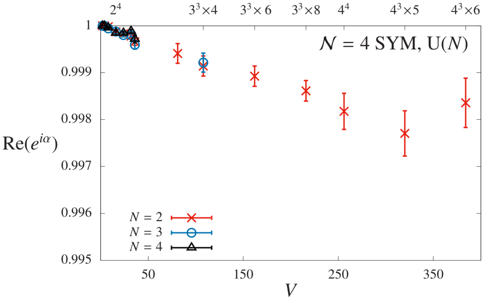
Complex pfaffian $\text{pf}\, \mathcal{D} = |\text{pf}\, \mathcal{D}| e^{i\alpha}$ complicates importance sampling

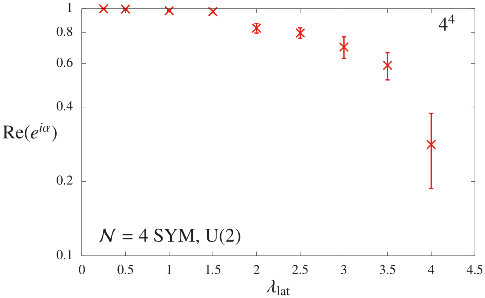
We phase quench, $\text{pf}\, \mathcal{D} \longrightarrow |\text{pf}\, \mathcal{D}|$, need to reweight $\langle \mathcal{O} \rangle = \dfrac{\left\langle \mathcal{O} e^{i\alpha} \right\rangle_{pq}}{\left\langle e^{i\alpha} \right\rangle_{pq}}$

$\mathcal{N} = 4$ SYM sign problem

Fix $\lambda_{\text{lat}} = g_{\text{lat}}^2 N = 0.5$

Pfaffian nearly real positive
for all accessible volumes





Fix 4^4 volume

Fluctuations increase with coupling

Signal-to-noise
becomes obstruction for $\lambda_{\text{lat}} \gtrsim 4$

Future: Lattice superQCD

[Catterall–Veernala, arXiv:1505.00467]

Preserve twisted supersymmetry sub-algebra in 2d or 3d

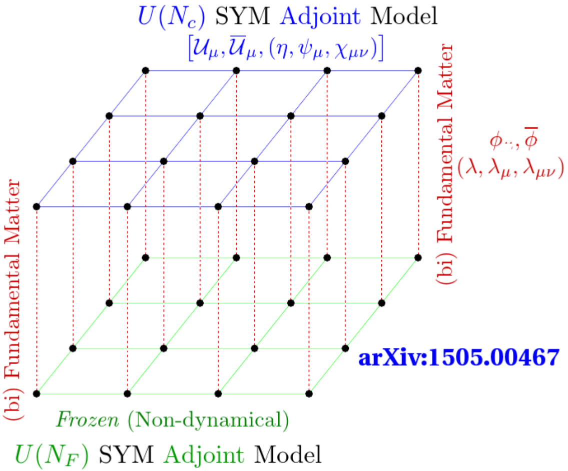
2-slice lattice SYM
with U(N) $\times$ U(F) gauge group

Adj. fields on each slice

Bi-fundamental in between

Decouple U(F) slice

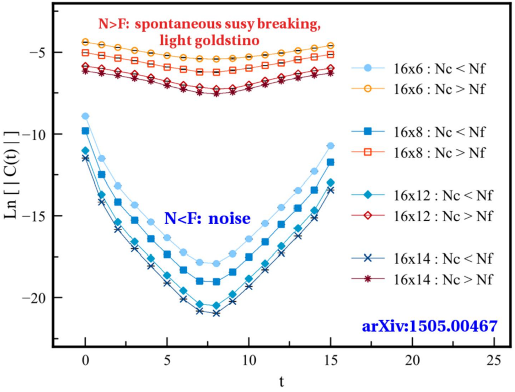
$\longrightarrow$ U(N) SQCD in $d-1$ dims.
with F fund. hypermultiplets

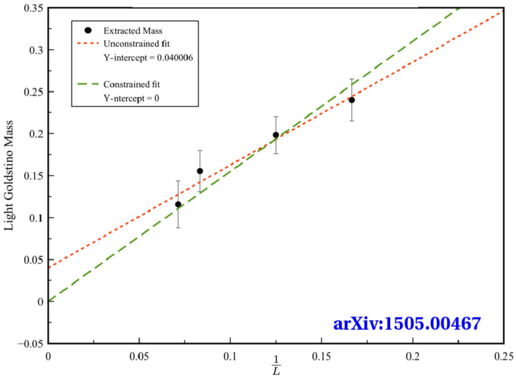


Dynamical susy breaking in 2d lattice superQCD

U(N) superQCD with F fundamental hypermultiplets

Spontaneous susy breaking requires $N > F$





Recap: An exciting time for lattice supersymmetry

✓ Preserve (some) susy in discrete space-time

$\longrightarrow$ practical lattice $\mathcal{N} = 4$ SYM, public code available

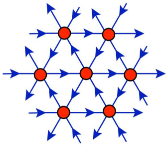
Reproduce reliable analytic results

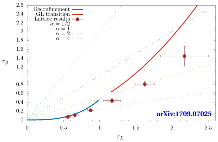
✓ 2d $\mathcal{N} = (8, 8)$ SYM thermodynamics consistent with holography

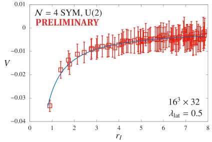
✓ Perturbative static potential Coulomb coefficient $C(\lambda)$

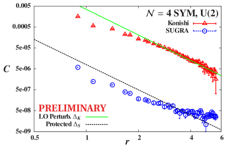
and Konishi operator conformal scaling dimension $\Delta_K(\lambda)$

Access new domains $\longrightarrow$ sign problem, lower-dim'l superQCD and more...









Thank you!

Collaborators

Simon Catterall, Raghav Jha, Toby Wiseman

also Georg Bergner, Poul Damgaard, Joel Giedt, Anosh Joseph

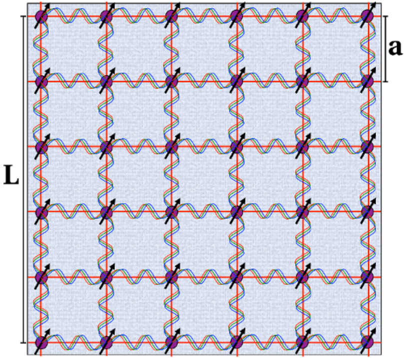
Funding and computing resources

UK Research and Innovation

Backup: Lattice field theory in a nutshell

Formally $\quad \langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \;\; \mathcal{O}(\Phi) \;\; e^{-S[\Phi]}$

Regularize by formulating theory in finite, discrete space-time $\longrightarrow$ **the lattice**



P. Vranas LLNL

Spacing between lattice sites ("a")
$\longrightarrow$ UV cutoff scale $1/a$

Remove cutoff: $\quad a \to 0 \quad (L/a \to \infty)$

Hypercubic $\longrightarrow$ automatic symmetries

Backup: Numerical lattice field theory calculations

High-performance computing
$\longrightarrow$ evaluate up to
$\sim$billion-dimensional integrals

Importance sampling Monte Carlo

Algorithms sample field configurations with probability $\frac{1}{\mathcal{Z}} e^{-S[\Phi]}$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \;\; \mathcal{O}(\Phi) \;\; e^{-S[\Phi]} \longrightarrow \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(\Phi_i) \;\; \text{with stat. uncertainty} \;\; \propto \frac{1}{\sqrt{N}}$$

Backup: Breakdown of Leibniz rule on the lattice

$\left\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\right\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = 2i\sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu$ is problematic

$\Longrightarrow$ try finite difference $\partial\phi(x) \longrightarrow \Delta\phi(x) = \frac{1}{a}\left[\phi(x+a) - \phi(x)\right]$

Crucial difference between ∂ and Δ

$$\begin{aligned}\Delta\left[\phi\eta\right] &= a^{-1}\left[\phi(x+a)\eta(x+a) - \phi(x)\eta(x)\right] \\ &= \left[\Delta\phi\right]\eta + \phi\Delta\eta + a\left[\Delta\phi\right]\Delta\eta\end{aligned}$$

Full supersymmetry requires Leibniz rule $\partial\left[\phi\eta\right] = \left[\partial\phi\right]\eta + \phi\partial\eta$

only recoverd in $a \to 0$ continuum limit for any local finite difference

Backup: Complexified gauge field from twisting

Combining A_μ and $\Phi^{\mathrm{I}} \longrightarrow \mathcal{A}_a$ and $\overline{\mathcal{A}}_a$

produces $\mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1)$ gauge theory

Complicates lattice action but needed so that $\mathcal{Q}\, \mathcal{A}_a = \psi_a$

Further motivation: Under $\mathrm{SO}(d)_{tw} = \mathrm{diag}\left[\mathrm{SO}(d)_{\mathrm{euc}} \otimes \mathrm{SO}(d)_R\right]$

$$\begin{aligned} A_\mu &\sim \text{vector} \otimes \text{scalar} &= \text{vector} \\ \Phi^{\mathrm{I}} &\sim \text{scalar} \otimes \text{vector} &= \text{vector} \end{aligned}$$

Easiest to see in 5d (then dimensionally reduce)

$$\mathcal{A}_a = A_a + i\Phi_a \longrightarrow (A_\mu, \phi) + i(\Phi_\mu, \overline{\phi})$$

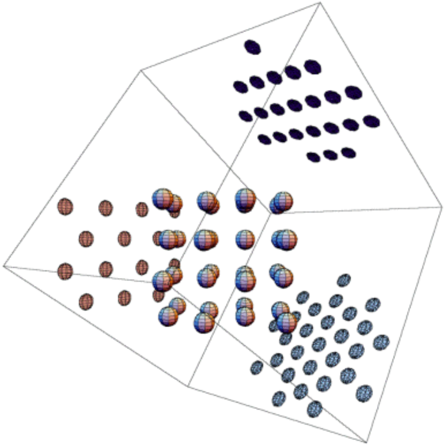
Backup: A_4^* lattice from five dimensions

Again dimensionally reduce, treating all five gauge links symmetrically

Start with hypercubic lattice
in 5d momentum space

Symmetric constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in real space

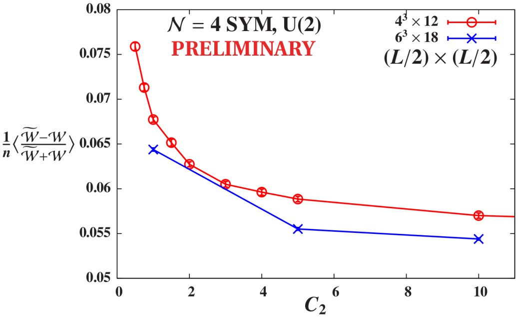


Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

$$\text{“}\mathcal{Q} \; + \; \text{discrete} \;\; R_a \subset \mathrm{SO}(4)_{tw} \;\; = \;\; \mathcal{Q}_a \;\; \text{and} \;\; \mathcal{Q}_{ab}\text{”}$$

Test R_a on Wilson loops $\widetilde{\mathcal{W}}_{ab} \equiv R_a \mathcal{W}_{ab}$

Tune coeff. c_2 of d^2 term
to ensure restoration in continuum



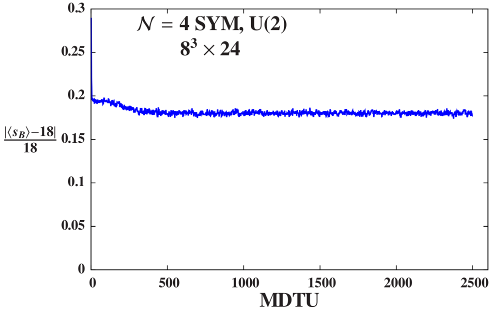
Backup: Problem with SU(N) flat directions

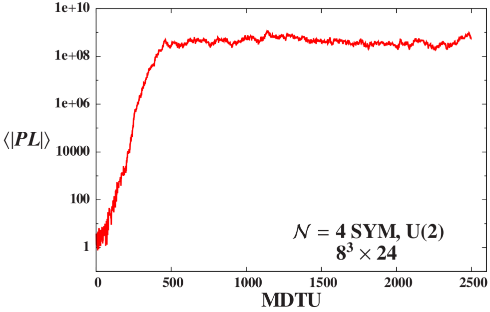
$\mu^2/\lambda_{\text{lat}}$ too small $\longrightarrow$ $\mathcal{U}_a$ can move far from continuum form $\mathbb{I}_N + \mathcal{A}_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 2.5$ on $8^3 \times 24$ volume

Left: Bosonic action stable $\sim$18% off its supersymmetric value

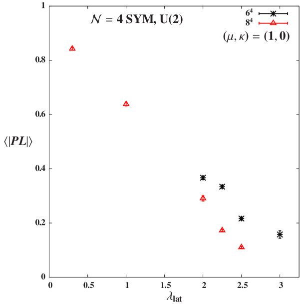
Right: (Complexified) Polyakov loop wanders off to $\sim 10^9$

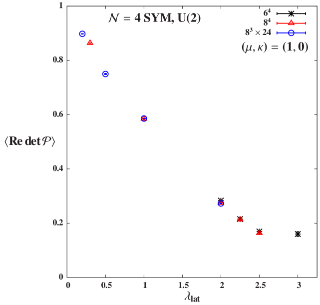


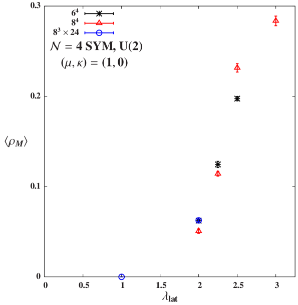


Backup: Problem with U(1) flat directions

Monopole condensation $\longrightarrow$ confined lattice phase not present in continuum







Around the same $2\lambda_{\text{lat}} \approx 2\ldots$

Left: Polyakov loop falls towards zero

Center: Plaquette determinant falls towards zero

Right: Density of U(1) monopole world lines becomes non-zero

Backup: Regulating SU(N) flat directions

Add soft $\mathcal{Q}$-breaking scalar potential to lattice action

$$S = \frac{N}{4\lambda_{\text{lat}}}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V\right]$$

$$V = \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2$$ lifts SU(N) flat directions,

ensures $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

Correct continuum limit requires $\mu^2 \to 0$ to restore $\mathcal{Q}$ and recover moduli space

Typically scale $\mu \propto 1/L$ in $L \to \infty$ continuum extrapolation

Backup: Poorly regulating U(1) flat directions

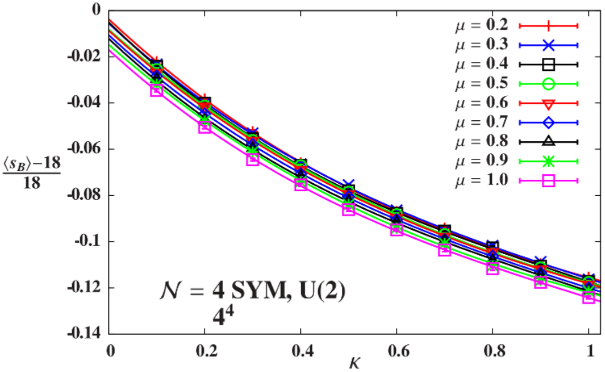
In earlier work we added another soft $\mathcal{Q}$-breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}}\mu^2 \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2 + \kappa \sum_{a<b} \left|\det \mathcal{P}_{ab} - 1\right|^2$$

More sensitivity to κ than to μ^2

Showing $\mathcal{Q}$ Ward identity from bosonic action

$$\langle s_B \rangle = 9N^2/2$$

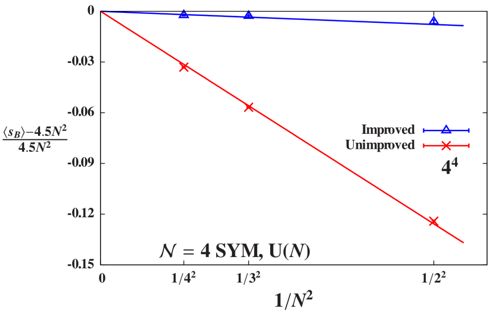


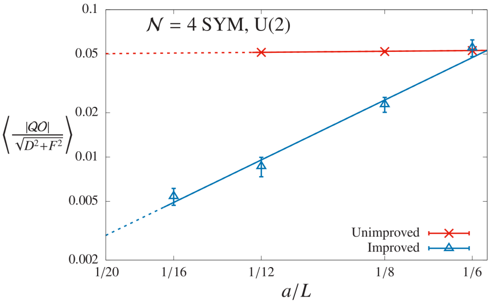
Backup: Better regulating U(1) flat directions

$$S = \frac{N}{4\lambda_{\text{lat}}}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\left\{\overline{\mathcal{D}}_a\mathcal{U}_a + G\sum_{a<b}\left[\det\mathcal{P}_{ab} - 1\right]\mathbb{I}_N\right\} - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V\right]$$

$\mathcal{Q}$ Ward identity violations scale $\propto 1/N^2$ (**left**) and $\propto (a/L)^2$ (**right**)

$\sim$ effective '$O(a)$ improvement' since $\mathcal{Q}$ forbids all dim-5 operators





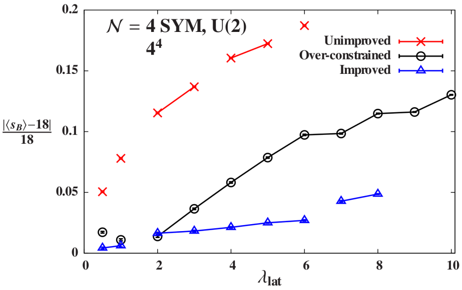
Backup: Supersymmetric moduli space modification [arXiv:1505.03135]

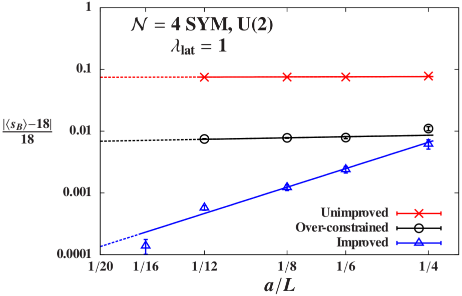
Method to impose $\mathcal{Q}$-invariant constraints on generic site operator $\mathcal{O}(n)$

Modify auxiliary field equations of motion $\longrightarrow$ moduli space

$$d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) \qquad \longrightarrow \qquad d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

However, both U(1) and SU(N) $\in \mathcal{O}(n)$ over-constrains system





Backup: Dimensional reduction to 2d $\mathcal{N} = (8, 8)$ SYM

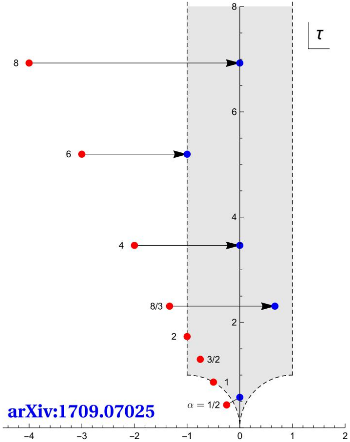
Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

$A_4^* \longrightarrow A_2^*$ (triangular) lattice

Torus **skewed** depending on $\alpha = N_t/L$

Modular transformation into fundamental domain
$\longrightarrow$ some skewed tori actually rectangular

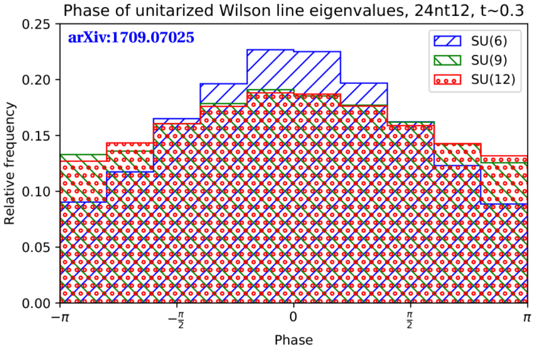
Also need to stabilize compactified links
to ensure broken center symmetries

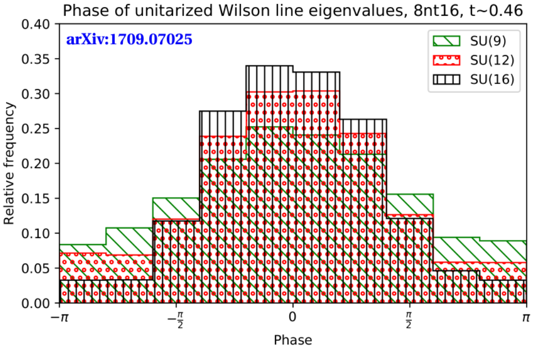


arXiv:1709.07025

Backup: 2d $\mathcal{N} = (8, 8)$ SYM Wilson line eigenvalues

Check 'spatial deconfinement' through Wilson line eigenvalue phases



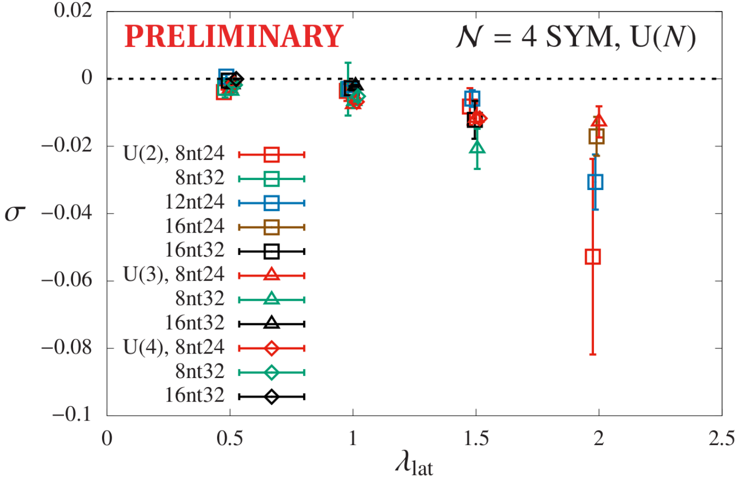


Left: $\alpha = 2$ distributions more extended as *N* increases $\longrightarrow$ D1 black string

Right: $\alpha = 1/2$ distributions more compact as *N* increases $\longrightarrow$ D0 black hole

Backup: Static potential is Coulombic at all λ

String tension σ from fits to confining form $V(r) = A - C/r + \sigma r$



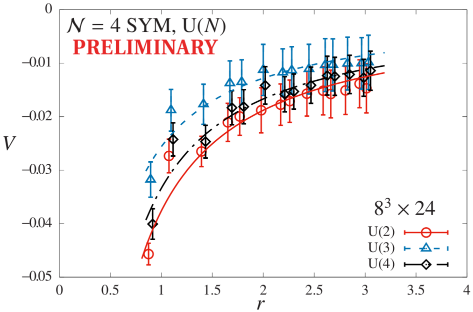
Slightly negative values flatten $V(r_I)$ for $r_I \lesssim L/2$

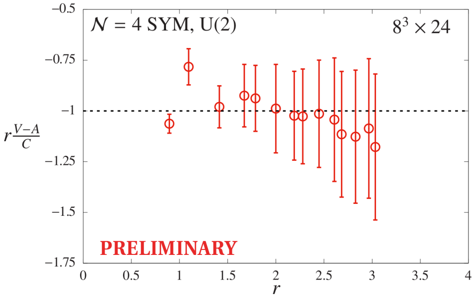
$\sigma \to 0$ as accessible range of r_I increases on larger volumes

Backup: Discretization artifacts in static potential

Discretization artifacts visible at short distances

where Coulomb term in $V(r) = A - C/r$ is most significant





Danger of distorting Coulomb coefficient C

Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential

Associate $V(r)$ data with r from Fourier transform of gluon propagator

Recall $\displaystyle \frac{1}{4\pi^2 r^2} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{e^{ir\cdot k}}{k^2}$ where $\displaystyle \frac{1}{k^2} = G(k)$ in continuum

$$A_4^* \text{ lattice} \longrightarrow \frac{1}{r_I^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4\widehat{k}}{(2\pi)^4} \frac{\cos\left(ir_I \cdot \widehat{k}\right)}{4\sum_{\mu=1}^{4} \sin^2\left(\widehat{k}\cdot\widehat{e}_\mu \,/\, 2\right)}$$

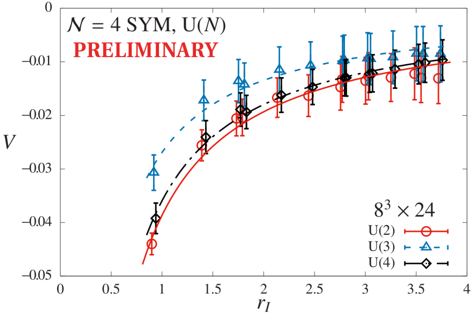
Tree-level lattice propagator from arXiv:1102.1725

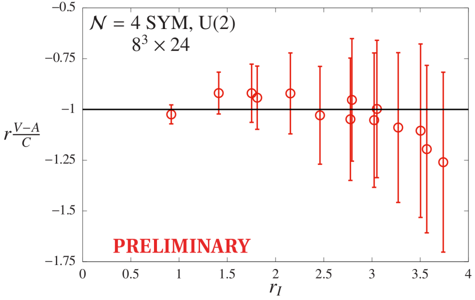
$\widehat{e}_\mu$ are A_4^* lattice basis vectors;

momenta $\widehat{k} = \frac{2\pi}{L}\sum_{\mu=1}^{4} n_\mu \widehat{g}_\mu$ depend on dual basis vectors

Backup: Tree-level-improved static potential

Significantly reduced discretization artifacts





Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $\mathcal{Q}$ and S_5 symmetries $\longleftrightarrow$ geometric structure

Simple transformation constructed in arXiv:1408.7067

$$\mathcal{U}'_a(n') = \xi\, \mathcal{U}_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) \qquad\qquad \eta'(n') = \eta(n)$$

$$\psi'_a(n') = \xi \left[\psi_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) + \mathcal{U}_a(n)\psi_a(n+\widehat{\mu}_a)\right] \qquad\qquad \text{etc.}$$

Doubles lattice spacing $a \longrightarrow a' = 2a$, with tunable rescaling factor ξ

Scalar fields from polar decomposition $\mathcal{U}(n) = e^{\varphi(n)}U(n)$

$\Longrightarrow$ shift $\varphi \longrightarrow \varphi + \log\xi$ to keep blocked U unitary

$\mathcal{Q}$-preserving RG transformation needed

to show only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Backup: Smearing for Konishi analyses

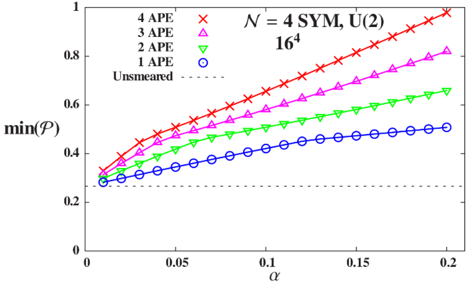
Smear to enlarge (MCRG or variational) operator basis

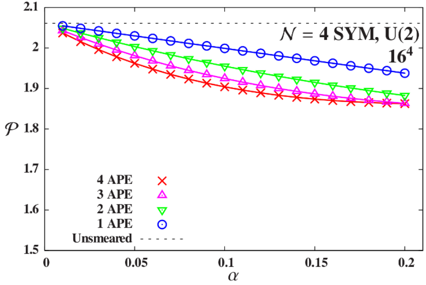
APE-like smearing: $\quad — \quad \longrightarrow \quad (1-\alpha)— \quad + \quad \frac{\alpha}{8}\sum \sqcap,$

staples built from unitary parts of links but no final unitarization

Average plaquette stable upon smearing (**right**),

minimum plaquette steadily increases (**left**)





Backup: More on dynamical susy breaking

Spontaneous susy breaking means $\langle 0 \left| H \right| 0 \rangle > 0$ or equivalently $\langle \mathcal{Q}\mathcal{O} \rangle \neq 0$

Twisted superQCD auxiliary field e.o.m. $\longleftrightarrow$ Fayet–Iliopoulos D-term potential

$$d = \overline{\mathcal{D}}_a \mathcal{U}_a + \sum_{i=1}^{F} \phi_i \overline{\phi}_i - r\mathbb{I}_N \qquad \longleftrightarrow \qquad \mathrm{Tr}\left[\left(\sum\nolimits_i \phi_i \overline{\phi}_i - r\mathbb{I}_N\right)^2\right] \in H$$

Have F scalar vevs to zero out N diagonal elements

$\longrightarrow N > F$ suggests susy breaking, $\langle 0 \left| H \right| 0 \rangle > 0 \longleftrightarrow \langle \mathcal{Q}\eta \rangle = \langle d \rangle \neq 0$