

Maximally supersymmetric Yang–Mills on the lattice

David Schaich (Bern)



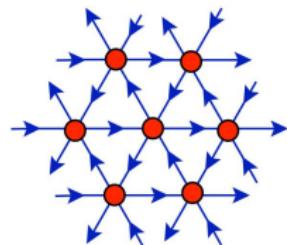
Swansea Theory Seminar
23 November 2018

[arXiv:1611.06561](https://arxiv.org/abs/1611.06561) [arXiv:1709.07025](https://arxiv.org/abs/1709.07025) [arXiv:1810.09282](https://arxiv.org/abs/1810.09282)

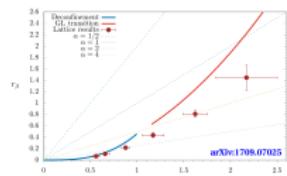
& more to come with Simon Catterall, Raghav Jha and Toby Wiseman

Overview and plan

WHY: Lattice supersymmetry



HOW: Lattice formulation highlights

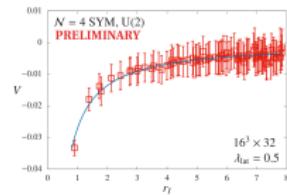


WHAT:

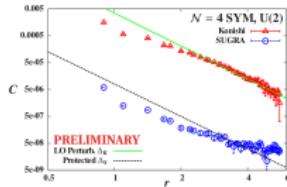
Dimensionally reduced (2d) thermodynamics

Static potential (4d)

Conformal scaling dimensions



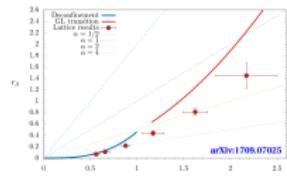
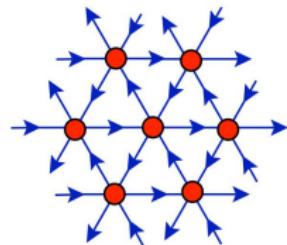
Prospects and future directions



Overview and plan

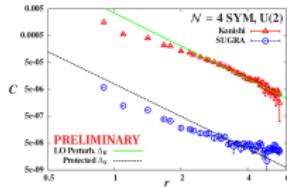
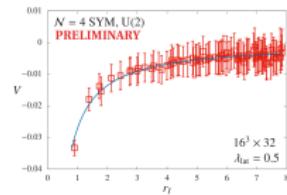
Central idea

Preserve (some) susy in discrete space-time
→ practical lattice investigations



Goals

- 1) Reproduce reliable results in perturbative, holographic, etc. regimes
- 2) Access new domains



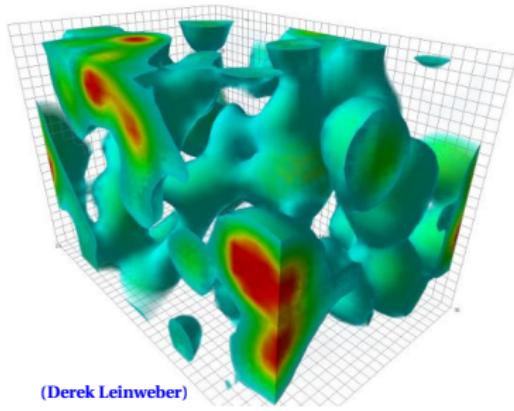
Motivations

Lattice field theory promises first-principles predictions
for strongly coupled supersymmetric QFTs

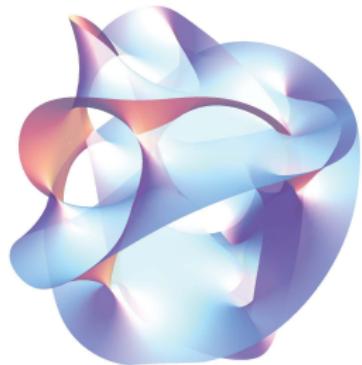
BSM



QFT



Holography



Obstruction

Supersymmetry is a space-time symmetry

4d Poincaré plus $4\mathcal{N}$ spinor generators Q_α^I and $\bar{Q}_{\dot{\alpha}}^I$ $I = 1, \dots, \mathcal{N}$

$$\left\{ Q_\alpha^I, \bar{Q}_{\dot{\alpha}}^J \right\} = 2\delta^{IJ} \sigma_{\alpha\dot{\alpha}}^\mu P_\mu \quad \text{broken in discrete space-time}$$

→ relevant susy-violating operators

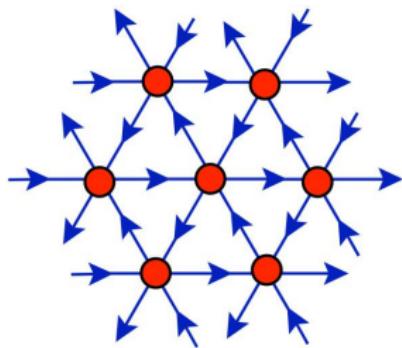


Solution

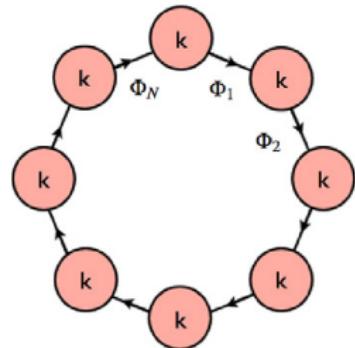
Preserve susy sub-algebra at non-zero lattice spacing

⇒ correct continuum limit with little or no fine tuning

Equivalent constructions from topological twisting and deconstruction



Review:
[arXiv:0903.4881](https://arxiv.org/abs/0903.4881)



Need 2^d supersymmetries in d dimensions

$d = 4 \rightarrow$ maximally supersymmetric Yang–Mills ($\mathcal{N} = 4$ SYM)

$\mathcal{N} = 4$ SYM in a nutshell

Arguably simplest non-trivial 4d QFT \rightarrow dualities, amplitudes, ...

$SU(N)$ gauge theory with $\mathcal{N} = 4$ fermions ψ^I and 6 scalars ϕ^{IJ} ,
all massless and in adjoint rep.

Symmetries relate coeffs of kinetic, Yukawa and ϕ^4 terms

Maximal 16 supersymmetries Q_α^I and $\bar{Q}_{\dot{\alpha}}^I$ $I = 1, \dots, 4$
transform under global $SU(4) \sim SO(6)$ R symmetry

Conformal \rightarrow β function is zero for any 't Hooft coupling $\lambda = g^2 N$

Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q_\alpha^1 & Q_\alpha^2 & Q_\alpha^3 & Q_\alpha^4 \\ \bar{Q}_{\dot{\alpha}}^1 & \bar{Q}_{\dot{\alpha}}^2 & \bar{Q}_{\dot{\alpha}}^3 & \bar{Q}_{\dot{\alpha}}^4 \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \bar{\mathcal{Q}} \gamma_5$$
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$

with $a, b = 1, \dots, 5$

R-symmetry index \times Lorentz index

$\Rightarrow \mathcal{Q}$ transform in reps of ‘twisted rotation group’

$$\mathrm{SO}(4)_{tw} \equiv \mathrm{diag} \left[\mathrm{SO}(4)_{\mathrm{euc}} \otimes \mathrm{SO}(4)_R \right] \qquad \qquad \mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$$

Change of variables $\rightarrow \mathcal{Q}$ with integer ‘spin’ under $\mathrm{SO}(4)_{tw}$

Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

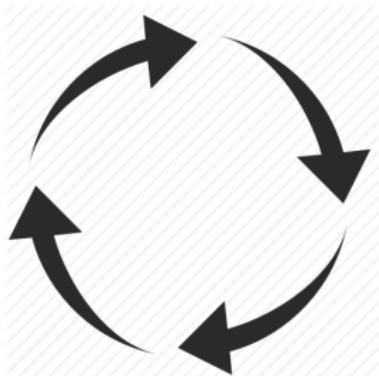
$$\begin{pmatrix} Q_\alpha^1 & Q_\alpha^2 & Q_\alpha^3 & Q_\alpha^4 \\ \bar{Q}_{\dot{\alpha}}^1 & \bar{Q}_{\dot{\alpha}}^2 & \bar{Q}_{\dot{\alpha}}^3 & \bar{Q}_{\dot{\alpha}}^4 \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \bar{\mathcal{Q}} \gamma_5$$
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$

with $a, b = 1, \dots, 5$

Discrete space-time

Can preserve closed sub-algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$



Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q_\alpha^1 & Q_\alpha^2 & Q_\alpha^3 & Q_\alpha^4 \\ \bar{Q}_{\dot{\alpha}}^1 & \bar{Q}_{\dot{\alpha}}^2 & \bar{Q}_{\dot{\alpha}}^3 & \bar{Q}_{\dot{\alpha}}^4 \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \bar{\mathcal{Q}} \gamma_5$$
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$

with $a, b = 1, \dots, 5$

Discrete space-time

Can preserve closed subalgebra



$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$

Completing the twist

Fields also transform with integer spin under $\text{SO}(4)_{tw}$ — no spinors

Ψ and $\bar{\Psi}$ $\rightarrow \eta, \psi_a$ and χ_{ab}

A_μ and Φ^I \rightarrow complexified gauge field \mathcal{A}_a and $\bar{\mathcal{A}}_a$
 $\rightarrow \text{U}(N) = \text{SU}(N) \otimes \text{U}(1)$ gauge theory

✓ \mathcal{Q} interchanges bosonic \longleftrightarrow fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\mathcal{Q} \mathcal{A}_a = \psi_a$$

$$\mathcal{Q} \psi_a = 0$$

$$\mathcal{Q} \chi_{ab} = -\bar{\mathcal{F}}_{ab}$$

$$\mathcal{Q} \bar{\mathcal{A}}_a = 0$$

$$\mathcal{Q} \eta = d$$

$$\mathcal{Q} d = 0$$



bosonic auxiliary field with e.o.m. $d = \bar{\mathcal{D}}_a \mathcal{A}_a$

Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking \mathcal{Q}_a and \mathcal{Q}_{ab}

Covariant derivatives \rightarrow finite difference operators

Complexified gauge fields $A_a \rightarrow$ gauge links $U_a \in \mathfrak{gl}(N, \mathbb{C})$

$$\mathcal{Q} A_a \rightarrow \mathcal{Q} U_a = \psi_a \quad \mathcal{Q} \psi_a = 0$$

$$\mathcal{Q} \chi_{ab} = -\bar{\mathcal{F}}_{ab} \quad \mathcal{Q} \bar{A}_a \rightarrow \mathcal{Q} \bar{U}_a = 0$$

$$\mathcal{Q} \eta = d \quad \mathcal{Q} d = 0$$

Geometry: η on sites, ψ_a on links, etc.

Susy lattice action ($\mathcal{Q}S = 0$) from $\mathcal{Q}^2 \cdot = 0$ and **Bianchi identity**

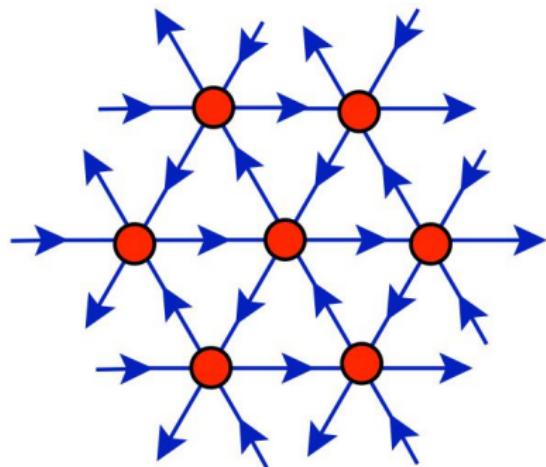
$$S = \frac{N}{4\lambda_{\text{lat}}} \text{Tr} \left[\mathcal{Q} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \bar{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \bar{D}_c \chi_{de} \right]$$

Five links in four dimensions $\longrightarrow A_4^*$ lattice

$A_4^* \sim$ 4d analog of 2d triangular lattice

Basis vectors linearly dependent
and non-orthogonal

Large S_5 point group symmetry



S_5 irreps precisely match onto irreps of twisted $SO(4)_{tw}$

$$\mathbf{5} = \mathbf{4} \oplus \mathbf{1} : \psi_a \longrightarrow \psi_\mu, \bar{\eta}$$

$$\mathbf{10} = \mathbf{6} \oplus \mathbf{4} : \chi_{ab} \longrightarrow \chi_{\mu\nu}, \bar{\psi}_\mu$$

$S_5 \longrightarrow SO(4)_{tw}$ in continuum limit restores \mathcal{Q}_a and \mathcal{Q}_{ab}

Checkpoint

Analytic results for twisted $\mathcal{N} = 4$ SYM on A_4^* lattice

$U(N)$ gauge invariance + \mathcal{Q} + S_5 lattice symmetries

- Moduli space preserved to all orders
- One-loop lattice β function vanishes
- Only one log. tuning to recover continuum \mathcal{Q}_a and \mathcal{Q}_{ab}

[[arXiv:1102.1725](#), [arXiv:1306.3891](#), [arXiv:1408.7067](#)]

Not quite suitable for numerical calculations

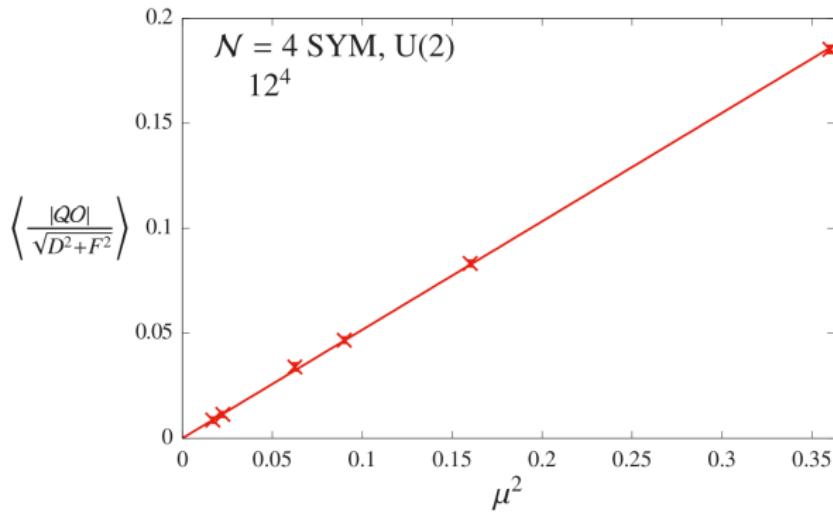
Must regulate zero modes and flat directions, especially in $U(1)$ sector

Two deformations stabilize lattice calculations

(i) Add $SU(N)$ scalar potential $\propto \mu^2 \sum_a (\text{Tr} [\mathcal{U}_a \bar{\mathcal{U}}_a] - N)^2$

Softly breaks susy $\rightarrow \mathcal{Q}$ -violating operators vanish $\propto \mu^2 \rightarrow 0$

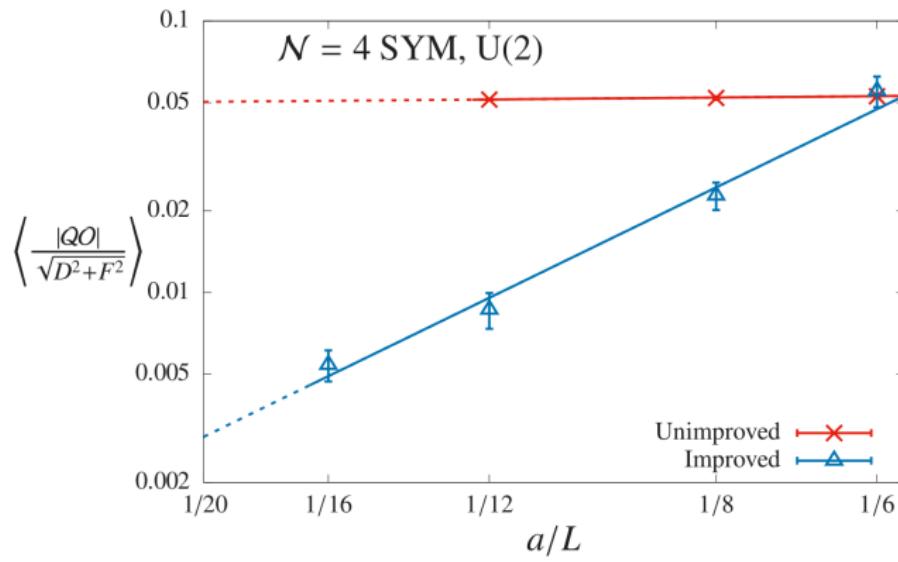
Test via Ward identity violations: $\mathcal{Q} [\eta \mathcal{U}_a \bar{\mathcal{U}}_a] \neq 0$



Two deformations stabilize lattice calculations

(ii) Constrain U(1) plaquette determinant $\sim G \sum_{a < b} (\det \mathcal{P}_{ab} - 1)$

Implemented supersymmetrically as Fayet–Iliopoulos D -term potential



Advertisement: Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$S_{\text{imp}} = S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \quad (18)$$

$$\begin{aligned} S'_{\text{exact}} &= \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr} \left[-\overline{\mathcal{F}}_{ab}(n) \mathcal{F}_{ab}(n) - \chi_{ab}(n) \mathcal{D}_{[a}^{(+)} \psi_{b]}(n) - \eta(n) \overline{\mathcal{D}}_a^{(-)} \psi_a(n) \right. \\ &\quad \left. + \frac{1}{2} \left(\overline{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) + G \sum_{a \neq b} (\det \mathcal{P}_{ab}(n) - 1) \mathbb{I}_N \right)^2 \right] - S_{\text{det}} \end{aligned}$$

$$S_{\text{det}} = \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr} [\eta(n)] \sum_{a \neq b} [\det \mathcal{P}_{ab}(n)] \text{Tr} [\mathcal{U}_b^{-1}(n) \psi_b(n) + \mathcal{U}_a^{-1}(n + \hat{\mu}_b) \psi_a(n + \hat{\mu}_b)]$$

$$S_{\text{closed}} = -\frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr} [\epsilon_{abcde} \chi_{de}(n + \hat{\mu}_a + \hat{\mu}_b + \hat{\mu}_c) \overline{\mathcal{D}}_c^{(-)} \chi_{ab}(n)],$$

$$S'_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_n \sum_a \left(\frac{1}{N} \text{Tr} [\mathcal{U}_a(n) \overline{\mathcal{U}}_a(n)] - 1 \right)^2$$

$\gtrsim 100$ inter-node data transfers in the fermion operator — non-trivial . . .

Public parallel code to reduce barriers to entry

github.com/daschaich/susy

Evolved from MILC QCD code, user guide in [arXiv:1410.6971](https://arxiv.org/abs/1410.6971)

(i) Thermodynamics on a 2-torus

arXiv:1709.07025

Dimensionally reduce to 2d $\mathcal{N} = (8, 8)$ SYM with four scalar Q

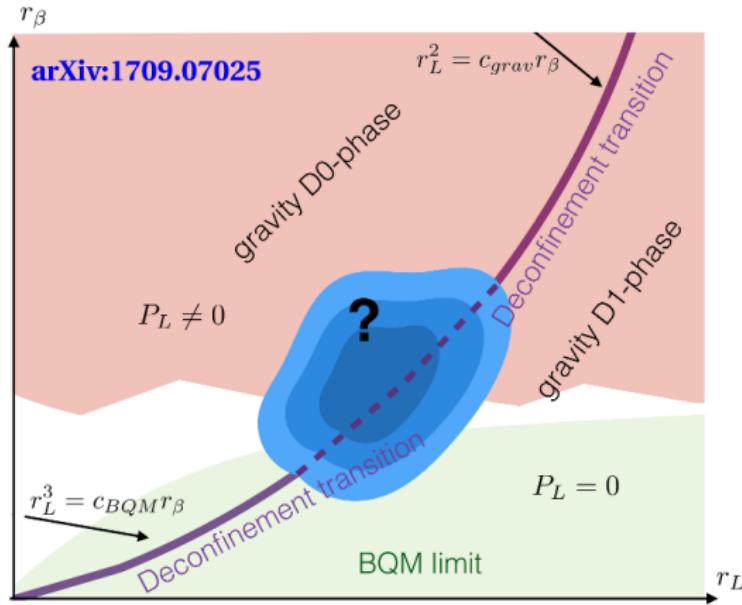
Low temperatures $t = 1/r_\beta \longleftrightarrow$ black holes in dual supergravity

For decreasing r_L at large N

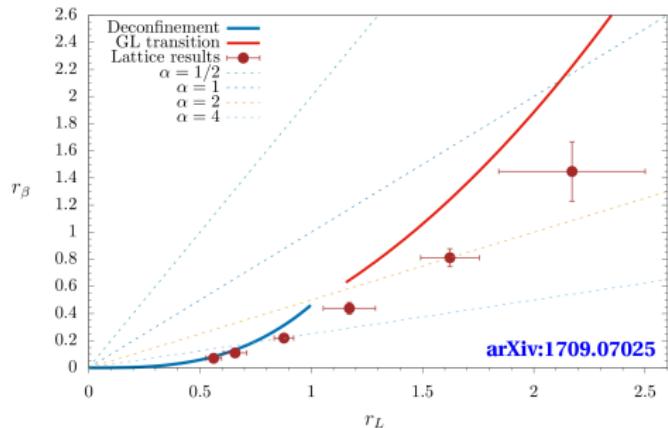
homogeneous black string (D1)
→ localized black hole (D0)



“spatial deconfinement”
signalled by Wilson line P_L



2d $\mathcal{N} = (8, 8)$ SYM lattice phase diagram results

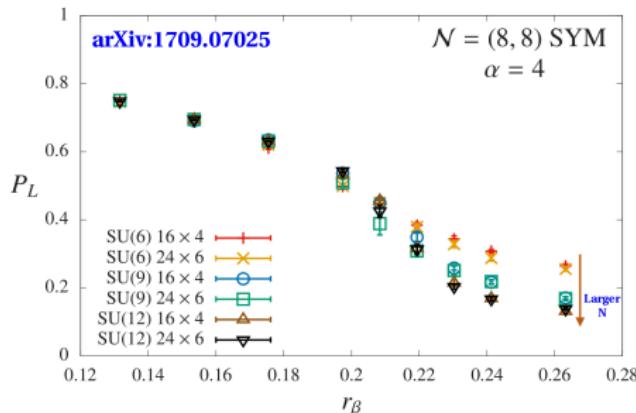


Good agreement
with high-temp. bosonic QM

Consistent with holography
at low temperatures

Example spatial deconfinement
transition in Wilson line

Fix aspect ratio $\alpha = r_L/r_\beta = 4$,
scan in $r_\beta = r_L/\alpha$



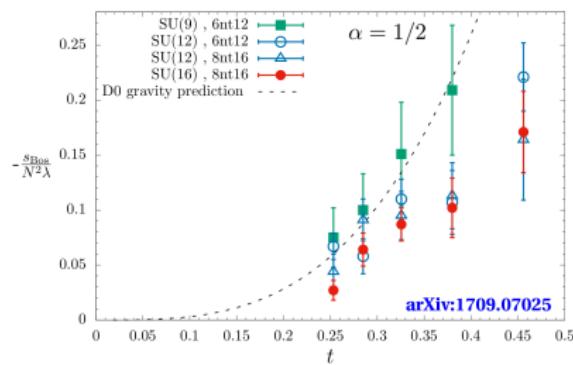
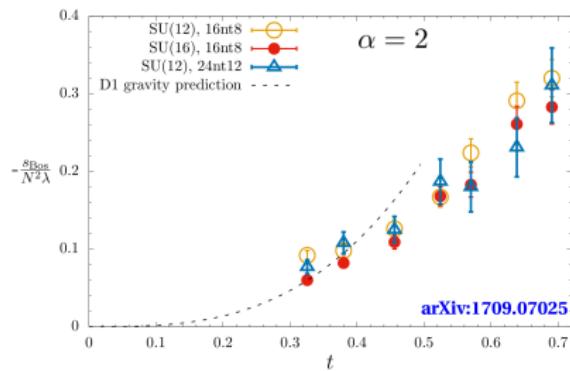
Dual black hole thermodynamics

Holography: bosonic action \longleftrightarrow dual black hole internal energy

$\propto t^3$ for large- r_L D1 phase

$\propto t^{3.2}$ for small- r_L D0 phase

Lattice results consistent with holography for sufficiently low $t \lesssim 0.4$

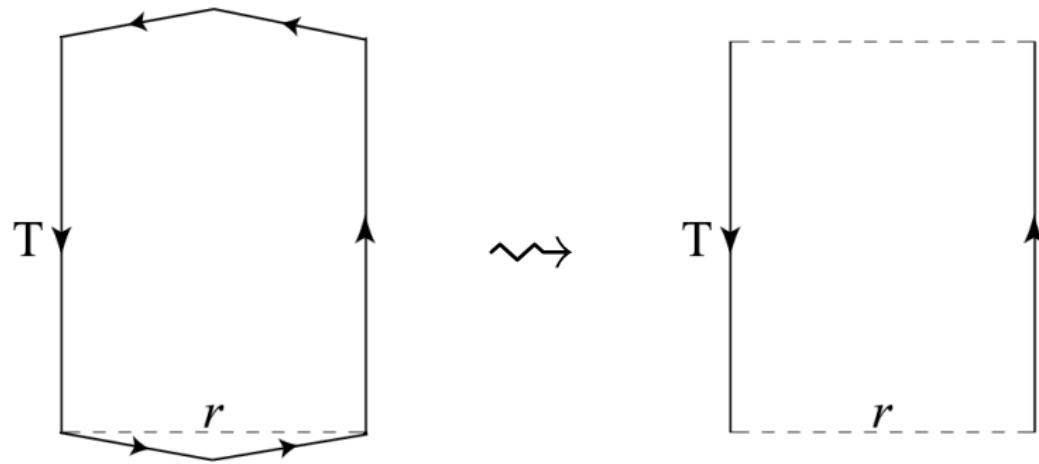


Need larger $N > 16$ to avoid instabilities at lower temperatures

(ii) 4d $\mathcal{N} = 4$ SYM static potential $V(r)$

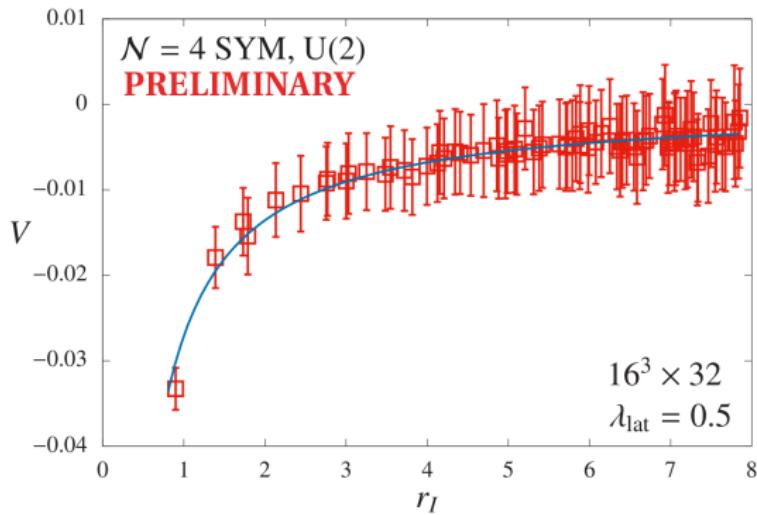
Static probes $\longrightarrow r \times T$ Wilson loops $W(r, T) \propto e^{-V(r) T}$

Coulomb gauge trick reduces A_4^* lattice complications



Static potential is Coulombic at all λ

Fits to confining $V(r) = A - C/r + \sigma r \rightarrow$ vanishing string tension σ
 \Rightarrow Fit to just $V(r) = A - C/r$ to extract Coulomb coefficient $C(\lambda)$

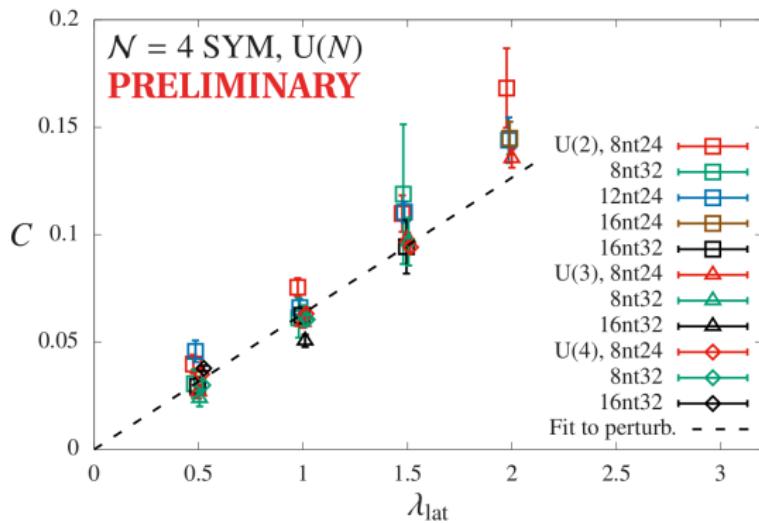


Discretization artifacts reduced by tree-level improved analysis

Coupling dependence of Coulomb coefficient

Continuum perturbation theory $\rightarrow C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

Holography $\rightarrow C(\lambda) \propto \sqrt{\lambda}$ for $N \rightarrow \infty$ and $\lambda \rightarrow \infty$ with $\lambda \ll N$



Consistent with leading-order perturbation theory for $\lambda_{\text{lat}} \leq 2$

(iii) Konishi operator scaling dimension

$\mathcal{O}_K(x) = \sum_I \text{Tr} [\Phi^I(x)\Phi^I(x)]$ is simplest conformal primary operator

Scaling dimension $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$ investigated through perturbation theory (& S duality), holography, conformal bootstrap

$$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

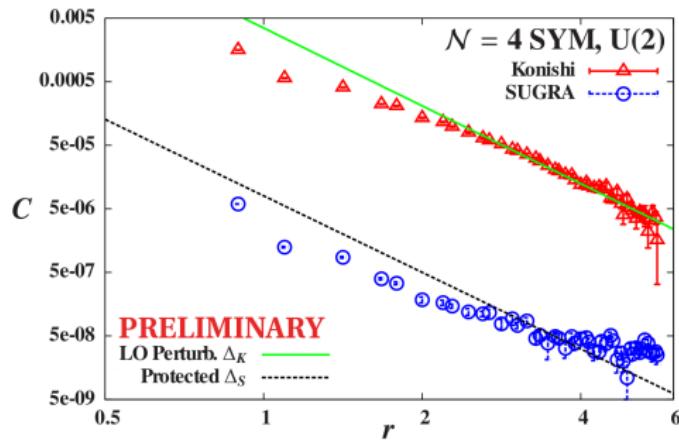
'SUGRA' is 20' op., $\Delta_S = 2$

Will compare:

Direct power-law decay

Finite-size scaling

Monte Carlo RG



(iii) Konishi operator scaling dimension

Lattice scalars $\varphi(n)$ from polar decomposition of complexified links

$$\mathcal{U}_a(n) \longrightarrow e^{\varphi_a(n)} U_a(n)$$

$$\mathcal{O}_K^{\text{lat}}(n) = \sum_a \text{Tr} [\varphi_a(n) \varphi_a(n)] - \text{vev}$$

$$\mathcal{O}_S^{\text{lat}}(n) \sim \text{Tr} [\varphi_a(n) \varphi_b(n)]$$

$$C_K(r) \equiv \mathcal{O}_K(x+r) \mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

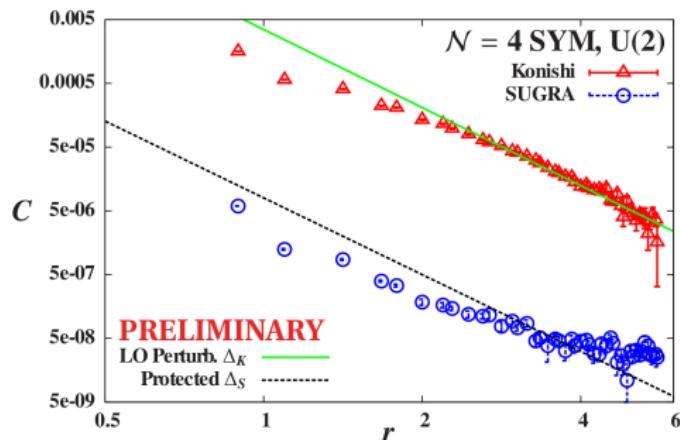
'SUGRA' is 20' op., $\Delta_S = 2$

Will compare:

Direct power-law decay

Finite-size scaling

Monte Carlo RG



Scaling dimensions from MCRG stability matrix

Lattice system: $H = \sum_i c_i \mathcal{O}_i$ (infinite sum)

Couplings flow under RG blocking $\rightarrow H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Fixed point $\rightarrow H^* = R_b H^*$ with couplings c_i^*

Linear expansion around fixed point \rightarrow **stability matrix** T_{ik}^*

$$c_i^{(n)} - c_i^* = \sum_k \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \Big|_{H^*} (c_k^{(n-1)} - c_k^*) \equiv \sum_k T_{ik}^* (c_k^{(n-1)} - c_k^*)$$

Correlators of $\mathcal{O}_i, \mathcal{O}_k \rightarrow$ elements of stability matrix [Swendsen, 1979]

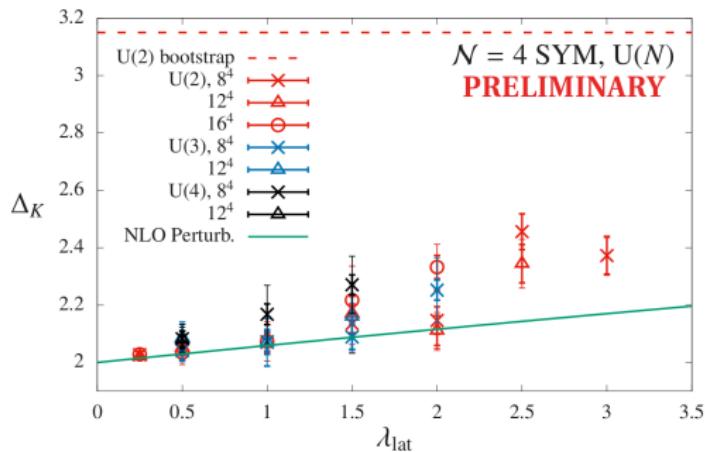
Eigenvalues of T_{ik}^* \rightarrow scaling dimensions of corresponding operators

Preliminary Δ_K results from Monte Carlo RG

Analyzing both $\mathcal{O}_K^{\text{lat}}$ and $\mathcal{O}_S^{\text{lat}}$

Imposing protected $\Delta_S = 2$
→ $\Delta_K(\lambda)$ looks perturbative

Systematic uncertainties from
different amounts of smearing



Complication from twisting $\text{SO}(4)_R \subset \text{SO}(6)_R$

$\mathcal{O}_K^{\text{lat}}$ mixes with $\text{SO}(4)_R$ -singlet part of $\text{SO}(6)_R$ -nonsinglet \mathcal{O}_S

Working on variational analyses to disentangle operators

Future: Pushing $\mathcal{N} = 4$ SYM to stronger coupling

✓ Reproduce reliable 4d results in perturbative regime

→ Check holographic predictions and access new domains

Sign problem seems to become obstruction

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}] [d\bar{\mathcal{U}}] \mathcal{O} e^{-S_B[\mathcal{U}, \bar{\mathcal{U}}]} \text{pf } \mathcal{D}[\mathcal{U}, \bar{\mathcal{U}}]$$

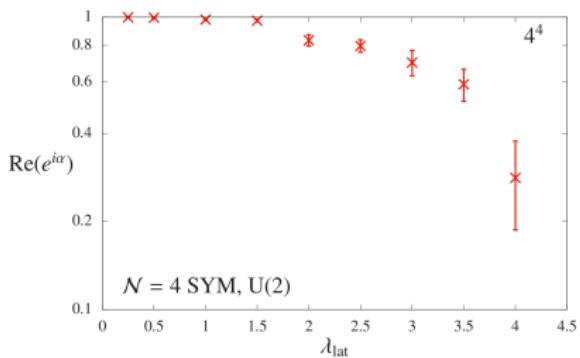
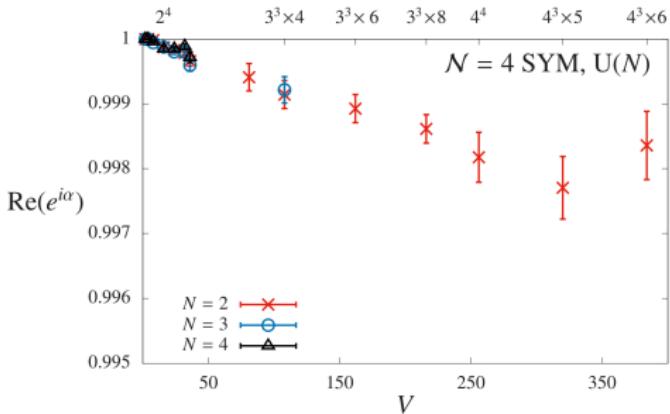
Complex pfaffian $\text{pf } \mathcal{D} = |\text{pf } \mathcal{D}| e^{i\alpha}$ complicates importance sampling

We phase quench, $\text{pf } \mathcal{D} \rightarrow |\text{pf } \mathcal{D}|$, need to reweight $\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}}$

$\mathcal{N} = 4$ SYM sign problem

Fix $\lambda_{\text{lat}} = g^2 N = 0.5$

Pfaffian nearly real positive
for all accessible volumes



Fix 4^4 volume

Fluctuations increase with coupling

Signal-to-noise
becomes obstruction for $\lambda_{\text{lat}} \gtrsim 4$

Future: Lattice superQCD (in 2d & 3d)

Preserve twisted supersymmetry sub-algebra

Proposed by Matsuura [0805.4491] and Sugino [0807.2683],

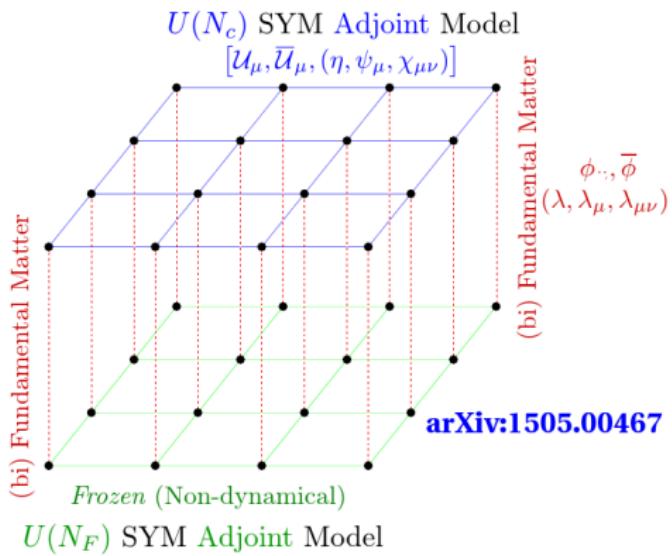
first numerical study by Catterall & Veernala [1505.00467]

2-slice lattice SYM
with $U(N) \times U(F)$ gauge group

Adj. fields on each slice

Bi-fundamental in between

Decouple $U(F)$ slice
→ $U(N)$ SQCD in $d - 1$ dims.
with F fund. hypermultiplets



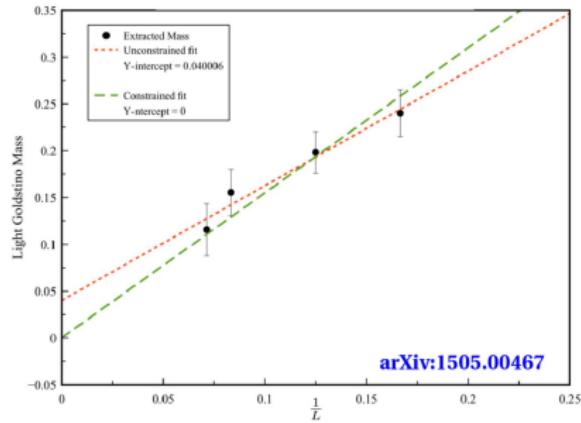
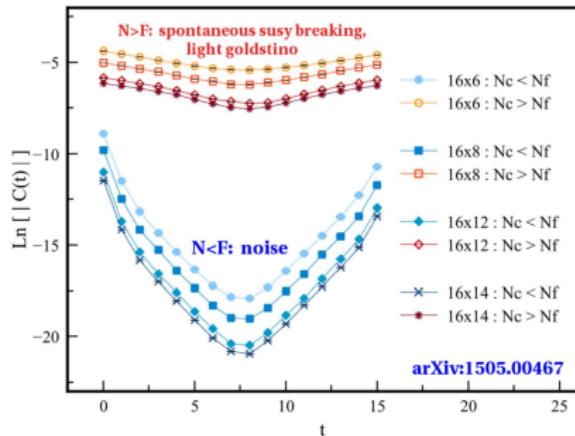
Dynamical susy breaking in 2d lattice superQCD

Auxiliary field e.o.m. \rightarrow Fayet–Iliopoulos D -term potential

$$d = \bar{\mathcal{D}}_a \mathcal{U}_a + \sum_{i=1}^F \phi_i \bar{\phi}_i - r \mathbb{I}_N \quad \rightarrow \quad S_D \propto \text{Tr} \left[\left(\sum_i \phi_i \bar{\phi}_i - r \mathbb{I}_N \right)^2 \right]$$

Zero out N diagonal elements via F scalar vevs

or else susy breaking, $\langle Q\eta \rangle = \langle d \rangle \neq 0 \longleftrightarrow \langle 0 | H | 0 \rangle > 0$



Recap: An exciting time for lattice supersymmetry

- ✓ Preserve (some) susy in discrete space-time

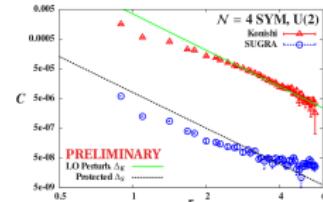
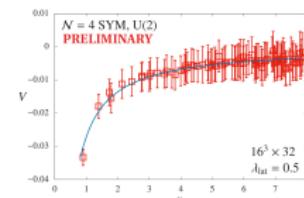
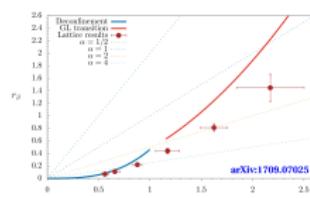
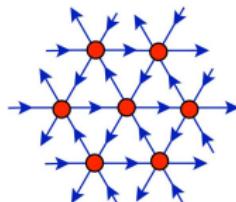
→ practical lattice $\mathcal{N} = 4$ SYM, **public code** available

Reproduce reliable analytic results

- ✓ 2d $\mathcal{N} = (8, 8)$ SYM thermodynamics consistent with holography
- ✓ Perturbative static potential Coulomb coefficient $C(\lambda)$ and Konishi operator conformal scaling dimension $\Delta_K(\lambda)$

Access new domains

- Tackling a sign problem at stronger couplings
- Lower-dimensional superQCD and more...



Thank you!

Collaborators

Simon Catterall, Raghav Jha, Toby Wiseman

also Georg Bergner, Poul Damgaard, Joel Giedt, Anosh Joseph

Funding and computing resources



Backup: Breakdown of Leibniz rule on the lattice

$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu = 2i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu$ is problematic

\implies try finite difference $\partial\phi(x) \longrightarrow \Delta\phi(x) = \frac{1}{a} [\phi(x+a) - \phi(x)]$

Crucial difference between ∂ and Δ

$$\begin{aligned}\Delta[\phi\eta] &= a^{-1} [\phi(x+a)\eta(x+a) - \phi(x)\eta(x)] \\ &= [\Delta\phi]\eta + \phi\Delta\eta + a[\Delta\phi]\Delta\eta\end{aligned}$$

Only recover Leibniz rule $\partial[\phi\eta] = [\partial\phi]\eta + \phi\partial\eta$ when $a \rightarrow 0$

\implies “discrete supersymmetry” breaks down on the lattice

Backup: Complexified gauge field from twisting

Combining A_μ and Φ^I \longrightarrow \mathcal{A}_a and $\bar{\mathcal{A}}_a$
 $\longrightarrow \text{U}(N) = \text{SU}(N) \otimes \text{U}(1)$ gauge theory

Complicates lattice action but needed so that $\mathcal{Q} \mathcal{A}_a = \psi_a$

Further motivation: Under $\text{SO}(d)_{tw} = \text{diag}[\text{SO}(d)_{\text{euc}} \otimes \text{SO}(d)_R]$

$$\begin{aligned} A_\mu &\sim \text{vector} \otimes \text{scalar} = \text{vector} \\ \Phi^I &\sim \text{scalar} \otimes \text{vector} = \text{vector} \end{aligned}$$

Easiest to see in 5d (then dimensionally reduce)

$$\mathcal{A}_a = A_a + i\Phi_a \longrightarrow (A_\mu, \phi) + i(\Phi_\mu, \bar{\phi})$$

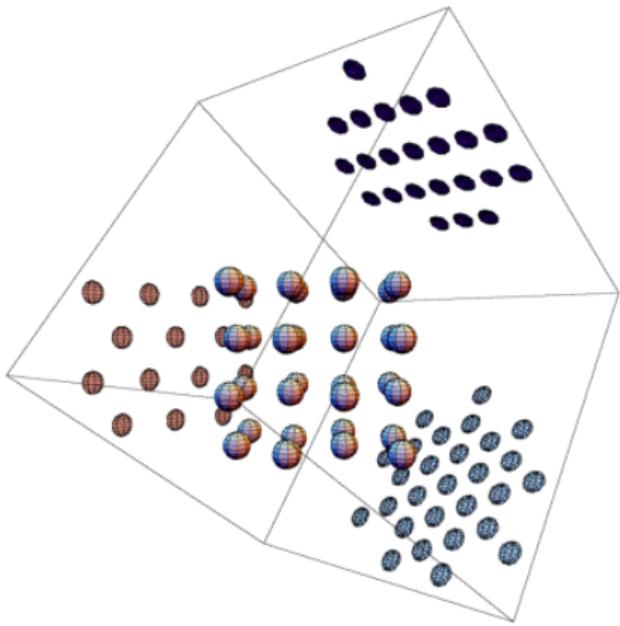
Backup: A_4^* lattice from five dimensions

Again dimensionally reduce, treating all five gauge links symmetrically

Start with hypercubic lattice
in 5d momentum space

Symmetric constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

Result is A_4 lattice
→ dual A_4^* lattice in real space

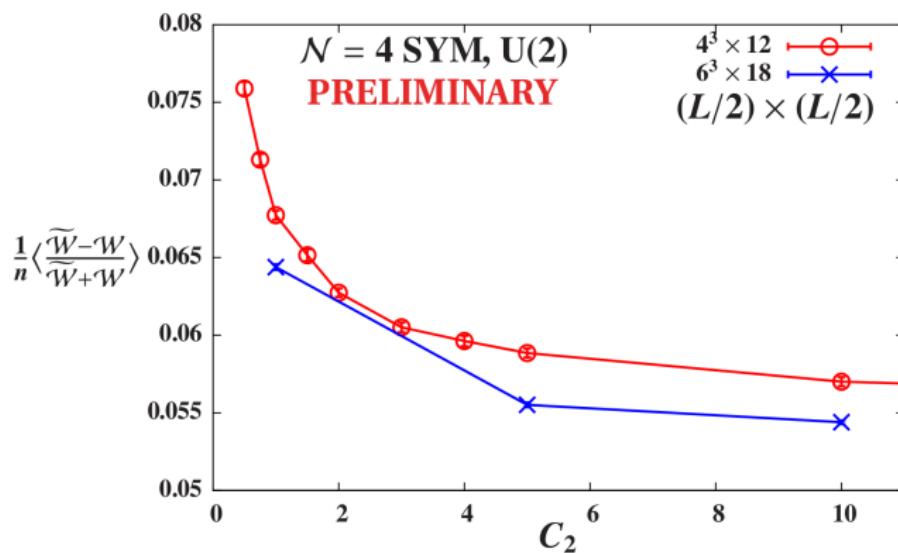


Backup: Restoration of \mathcal{Q}_a and \mathcal{Q}_{ab} supersymmetries

“ \mathcal{Q} + discrete $R_a \subset \text{SO}(4)_{\text{tw}}$ = \mathcal{Q}_a and \mathcal{Q}_{ab} ”

Test R_a on Wilson loops $\widetilde{\mathcal{W}}_{ab} \equiv R_a \mathcal{W}_{ab}$,

tune coeff. c_2 of d^2 term to ensure restoration in continuum



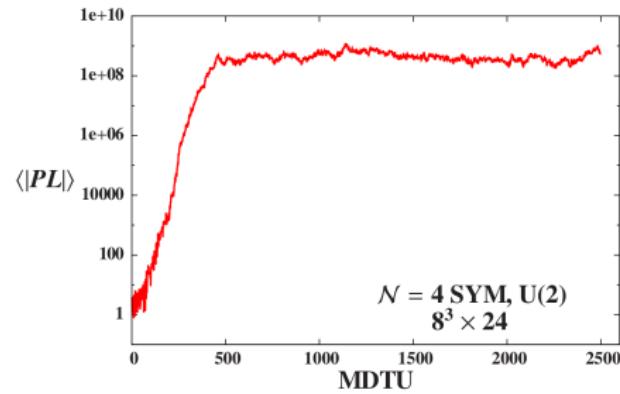
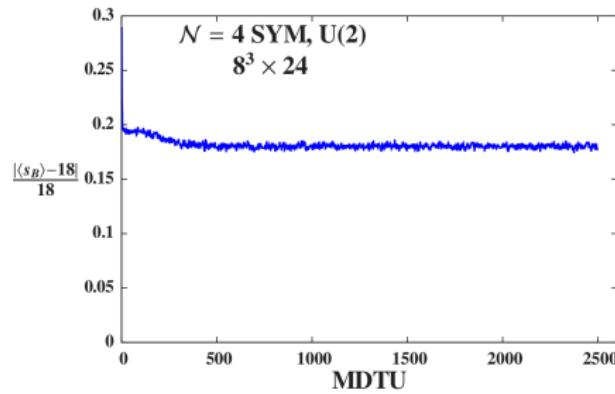
Backup: Problem with $SU(N)$ flat directions

$\mu^2/\lambda_{\text{lat}}$ too small $\rightarrow \mathcal{U}_a$ can move far from continuum form $\mathbb{I}_N + \mathcal{A}_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 2.5$ on $8^3 \times 24$ volume

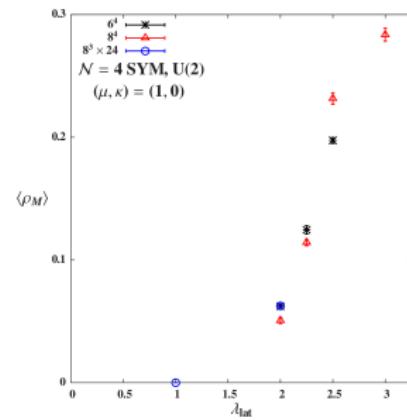
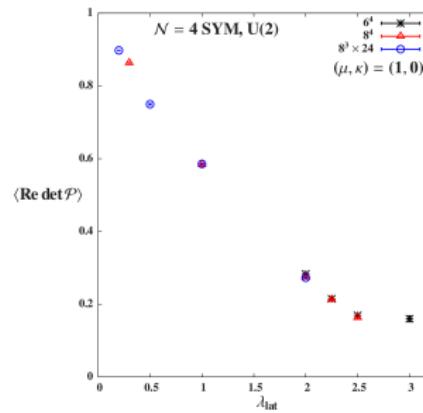
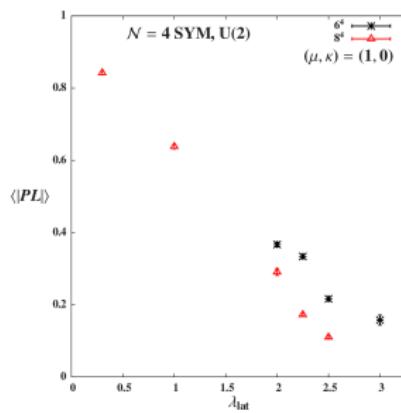
Left: Bosonic action stable $\sim 18\%$ off its supersymmetric value

Right: (Complexified) Polyakov loop wanders off to $\sim 10^9$



Backup: Problem with U(1) flat directions

Monopole condensation \rightarrow confined lattice phase
not present in continuum $\mathcal{N} = 4$ SYM



Around the same $2\lambda_{\text{lat}} \approx 2\dots$

Left: Polyakov loop falls towards zero

Center: Plaquette determinant falls towards zero

Right: Density of U(1) monopole world lines becomes non-zero

Backup: Regulating SU(N) flat directions

Add soft \mathcal{Q} -breaking scalar potential to lattice action

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[\mathcal{Q} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \bar{\mathcal{D}}_a \mathcal{U}_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \bar{\mathcal{D}}_c \chi_{de} + \mu^2 V \right]$$

$$V = \sum_a \left(\frac{1}{N} \text{Tr} [\mathcal{U}_a \bar{\mathcal{U}}_a] - 1 \right)^2$$

lifts SU(N) flat directions,
ensures $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

Correct continuum limit requires $\mu^2 \rightarrow 0$
to restore \mathcal{Q} and recover physical flat directions

Typically scale $\mu \propto 1/L$ in $L \rightarrow \infty$ continuum extrapolation

Backup: Poorly regulating U(1) flat directions

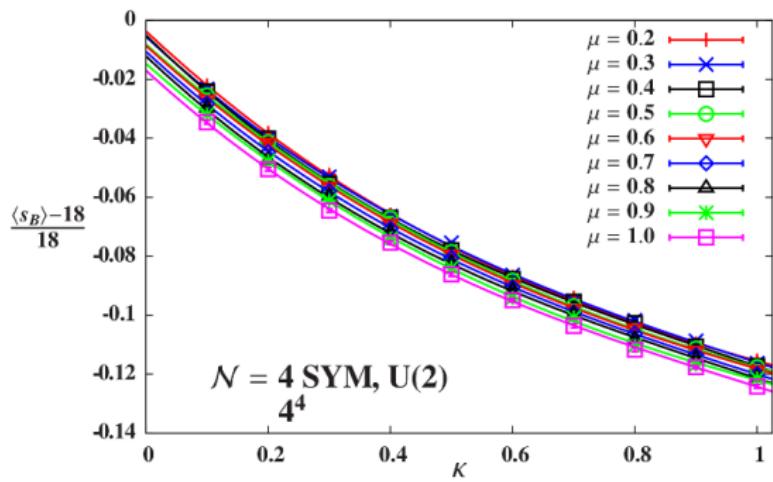
Until 2015 we added another soft \mathcal{Q} -breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_a \left(\frac{1}{N} \text{Tr} [\mathcal{U}_a \bar{\mathcal{U}}_a] - 1 \right)^2 + \kappa \sum_{a < b} |\det \mathcal{P}_{ab} - 1|^2$$

More sensitivity to κ
than to μ^2

Showing \mathcal{Q} Ward identity
from bosonic action

$$\langle s_B \rangle = 9N^2/2$$

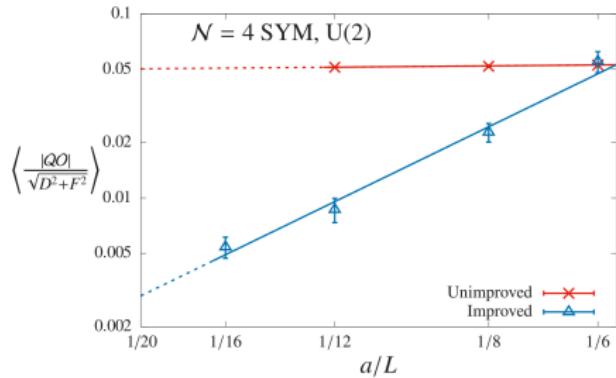
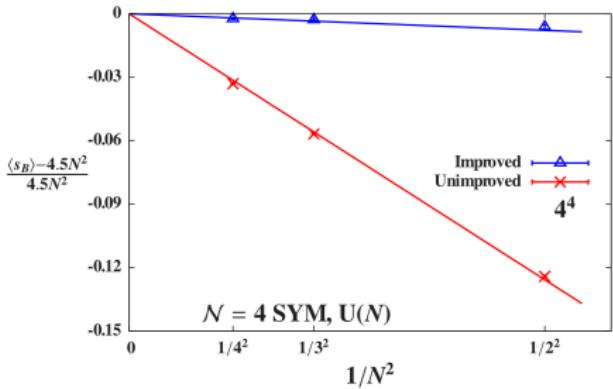


Backup: Better regulating U(1) flat directions

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[\mathcal{Q} \left(\chi_{ab} \mathcal{F}_{ab} + \downarrow - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \bar{\mathcal{D}}_c \chi_{de} + \mu^2 V \right]$$

$$\eta \left\{ \bar{\mathcal{D}}_a \mathcal{U}_a + G \sum_{a < b} [\det \mathcal{P}_{ab} - 1] \mathbb{I}_N \right\}$$

\mathcal{Q} Ward identity violations scale $\propto 1/N^2$ (**left**) and $\propto (a/L)^2$ (**right**)
 ~ effective ‘ $O(a)$ improvement’ since \mathcal{Q} forbids all dim-5 operators



Backup: Supersymmetric moduli space modification

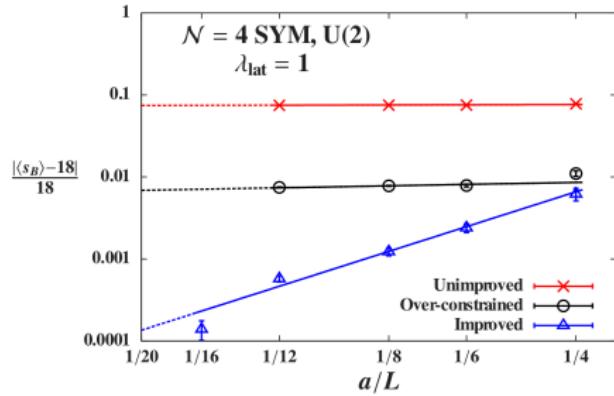
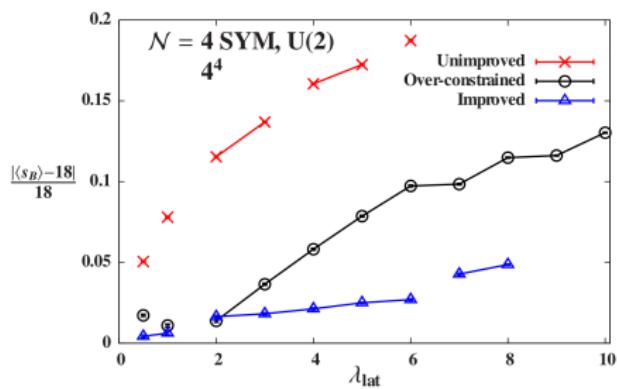
Method to impose \mathcal{Q} -invariant constraints

applicable to generic site operator $\mathcal{O}(n)$ [arXiv:1505.03135]

Modify auxiliary field equations of motion \rightarrow moduli space

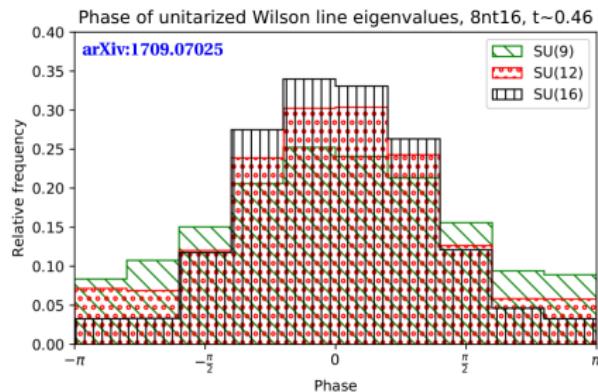
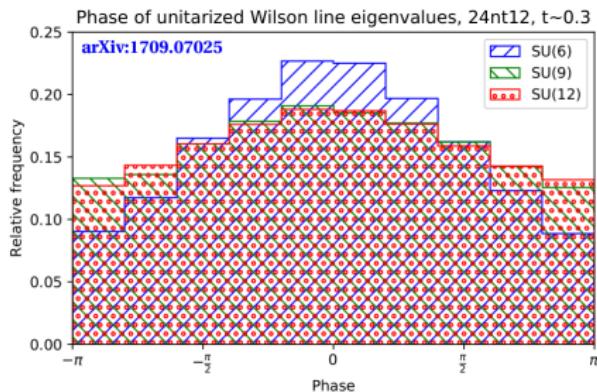
$$d(n) = \overline{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) \quad \rightarrow \quad d(n) = \overline{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

However, both $U(1)$ and $SU(N) \in \mathcal{O}(n)$ over-constrains system



Backup: $\mathcal{N} = (8, 8)$ SYM Wilson line eigenvalues

Check ‘spatial deconfinement’ through Wilson line eigenvalue phases



Left: $\alpha = 2$ distributions more extended as N increases

→ dual gravity describes homogeneous black string (D1 phase)

Right: $\alpha = 1/2$ distributions more compact as N increases

→ dual gravity describes localized black hole (D0 phase)

Backup: Dimensional reduction to 2d $\mathcal{N} = (8, 8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

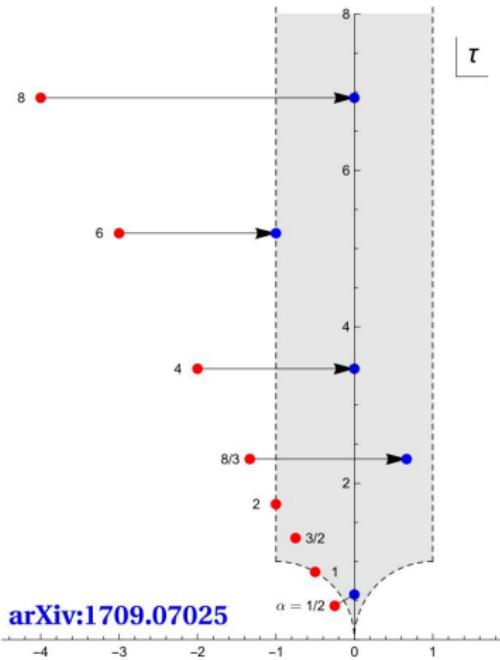
$A_4^* \longrightarrow A_2^*$ (triangular) lattice

Torus **skewed** depending on $\alpha = N_t/L$

Modular trans. into fund. domain

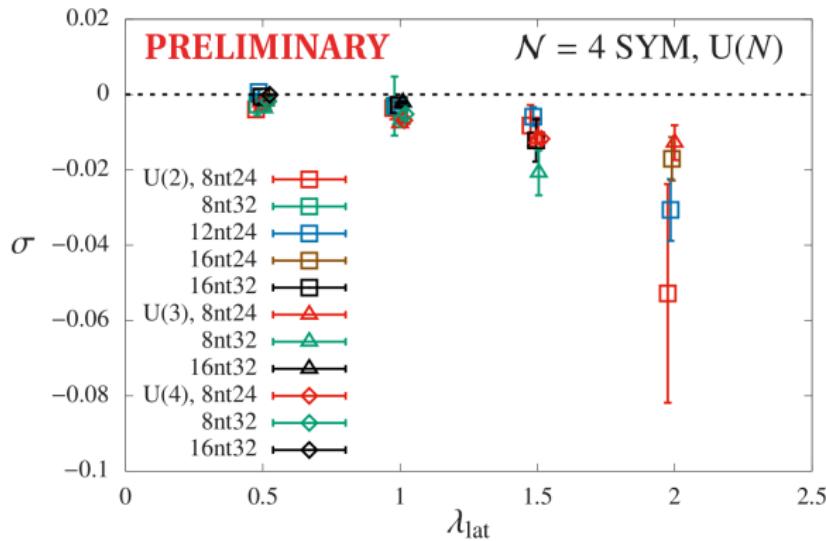
→ some skewed tori actually rectangular

Also need to stabilize compactified links
to ensure broken center symmetries



Backup: Static potential is Coulombic at all λ

String tension σ from fits to confining form $V(r) = A - C/r + \sigma r$



Slightly negative values flatten $V(r_l)$ for $r_l \lesssim L/2$

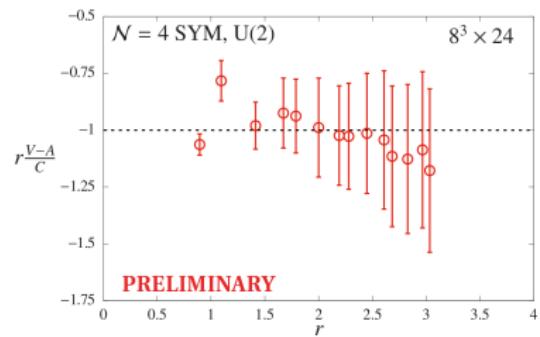
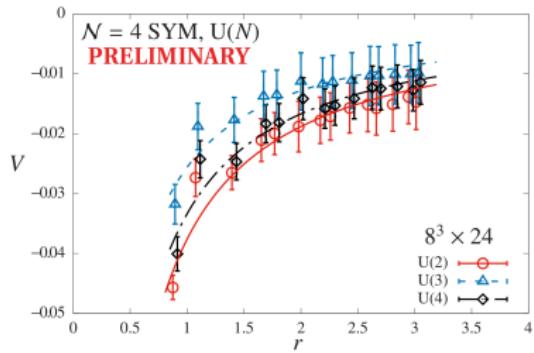
$\sigma \rightarrow 0$ as accessible range of r_l increases on larger volumes

Backup: Discretization artifacts in static potential

Discretization artifacts visible at short distances

where Coulomb term in $V(r) = A - C/r$ is most significant

Right: Fluctuations around Coulomb fit highlight artifacts



Danger of distorting Coulomb coefficient C

Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential

(Lang & Rebbi '82; Sommer '93; Necco '03)

Associate $V(r)$ data with r from Fourier transform of gluon propagator

Recall $\frac{1}{4\pi^2 r^2} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{e^{ir \cdot k}}{k^2}$ where $\frac{1}{k^2} = G(k)$ in continuum

$$A_4^* \text{ lattice} \longrightarrow \frac{1}{r_I^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \frac{\cos(ir_I \cdot \hat{k})}{4 \sum_{\mu=1}^4 \sin^2(\hat{k} \cdot \hat{e}_\mu / 2)}$$

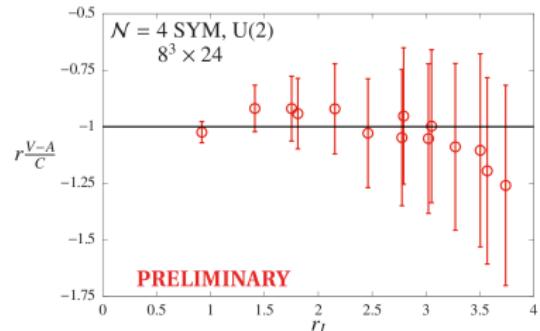
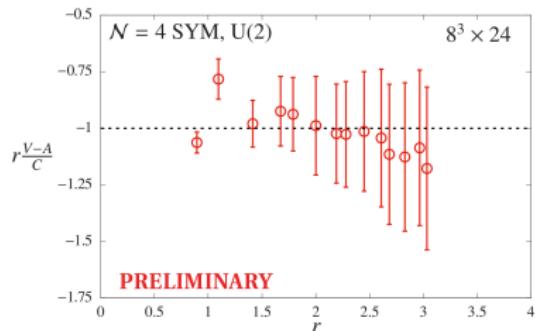
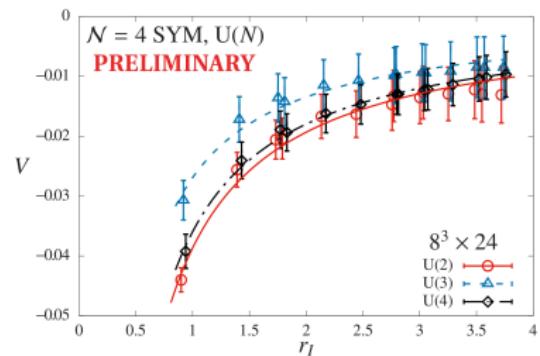
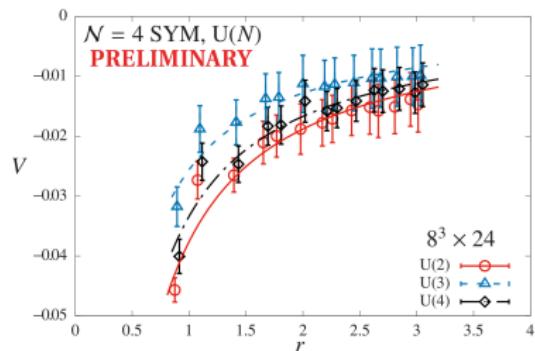
Tree-level lattice propagator from [arXiv:1102.1725](https://arxiv.org/abs/1102.1725)

\hat{e}_μ are A_4^* lattice basis vectors;

momenta $\hat{k} = \frac{2\pi}{L} \sum_{\mu=1}^4 n_\mu \hat{g}_\mu$ depend on dual basis vectors

Backup: Tree-level-improved static potential

Tree-level improvement significantly reduces discretization artifacts



Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve \mathcal{Q} and S_5 symmetries \longleftrightarrow geometric structure

Simple transformation constructed in [arXiv:1408.7067](https://arxiv.org/abs/1408.7067)

$$\begin{aligned}\mathcal{U}'_a(n') &= \xi \mathcal{U}_a(n) \mathcal{U}_a(n + \hat{\mu}_a) & \eta'(n') &= \eta(n) \\ \psi'_a(n') &= \xi [\psi_a(n) \mathcal{U}_a(n + \hat{\mu}_a) + \mathcal{U}_a(n) \psi_a(n + \hat{\mu}_a)] & \text{etc.}\end{aligned}$$

Doubles lattice spacing $a \rightarrow a' = 2a$, with tunable rescaling factor ξ

Scalar fields from polar decomposition $\mathcal{U}(n) = e^{\varphi(n)} U(n)$
→ shift $\varphi \rightarrow \varphi + \log \xi$ to keep blocked U unitary

This \mathcal{Q} -preserving RG transformation needed
to show only one log. tuning to recover continuum \mathcal{Q}_a and \mathcal{Q}_{ab}

Backup: Smearing for Konishi analyses

Smear to enlarge (MCRG or variational) operator basis

APE-like smearing: $\square \rightarrow (1 - \alpha)\square + \frac{\alpha}{8} \sum \square$,

staples built from unitary parts of links but no final unitarization
(unitarized smearing — e.g. stout — doesn't affect Konishi)

Average plaquette stable upon smearing (**right**),
minimum plaquette steadily increases (**left**)

