Lower-dimensional lattice supersymmetry

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Overview and plan

- Presentation goal: Survey of recent work on maximally supersymmetric Yang–Mills (SYM) theories in d < 4 dimensions
- **Research goal:** Reproduce known results in perturb., holographic, etc. regimes then use lattice to access new domains
- Review lattice supersymmetry and (4d) twisted lattice $\mathcal{N} = 4$ SYM
- 1d SYM bosonic action (others' work, 2007 through arXiv:1606.04951)
- 2d $\mathcal{N} = (8, 8)$ SYM phase diagram and bosonic action (arXiv:1709.07025)
- Work in progress: 1d supersymmetric mass deformation; max-SYM in 3d and 4d

Quick review of 4d lattice $\mathcal{N} = 4$ SYM

- 16 spinor generators ('supercharges') Q^A_{α} and $\overline{Q}^A_{\dot{\alpha}}$ with $A = 1, \dots, \mathcal{N}$ $\left\{Q^A_{\alpha}, \overline{Q}^B_{\dot{\alpha}}\right\} = 2\delta^{AB}\sigma^{\mu}_{\alpha\dot{\alpha}}P_{\mu} \longrightarrow$ supersymmetry algebra broken on the lattice
- Two ways to avoid impractical fine-tuning (will use both):
 - 1) Work in lower dimensions where theories are super-renormalizable
 - 2) Preserve closed sub-algebra of supersymmetries via topological twisting
- Topological twisting: (introduced for curved manifolds) $Q^{A}_{\alpha}, \overline{Q}^{B}_{\dot{\alpha}} \longrightarrow \mathcal{Q}, \mathcal{Q}_{\mu}, \mathcal{Q}_{\mu\nu}, \overline{\mathcal{Q}}_{\mu}, \overline{\mathcal{Q}}$ in integer-spin reps of "twisted rotation group" $\mathrm{SO}(4)_{tw} \equiv \mathrm{diag} \left[\mathrm{SO}(4)_{\mathrm{euc}} \otimes \mathrm{SO}(4)_{R} \right] \quad \text{with} \quad \mathrm{SO}(4)_{R} \subset \mathrm{SO}(6)_{R}$
- More generally, for $2 \le d \le 5$ $\operatorname{SO}(d)_{tw} \equiv \operatorname{diag} \left[\operatorname{SO}(d)_{\operatorname{euc}} \otimes \operatorname{SO}(d)_R \right]$ $Q \ge 2^d$ supercharges $\longrightarrow \lfloor Q/2^d \rfloor \ge 1$ closed susy subalgebras $\mathcal{Q}^2 = 0$
- Reducing 10-dim. $\mathcal{N} = 1$ SYM to d dims. \longrightarrow SO(10 d) R symmetry Fields: 16 fermions, d-component gauge field and 10 - d scalars,

all massless and in adjoint rep

• For $2 \le d \le 4$, discretize on A_d^{\star} lattice with d + 1 basis vectors (familiar triangular lattice for d = 2)

"D0 brane" SYM quantum mechanics

• Reduced to the point that twisting both impossible and unnecessary Only temporal component of gauge field remains, plus 9 scalars X^A

$$S_0 = \frac{N}{2\lambda} \int dt \, \operatorname{Tr}\left[\left(D_t X^A \right)^2 + \Psi^\alpha D_t \Psi^\alpha + \frac{1}{2} \left[X^A, X^B \right]^2 + i \Psi^\alpha \gamma^A_{\alpha\beta} \left[\Psi^\beta, X^A \right] \right]$$

with $A, B = 1, \cdots, 9$ and $\alpha, \beta = 1, \cdots, 16$

- Finite-temperature system holographically dual to stringy black hole geometry
- Temperature and dimension-3 't Hooft coupling \longrightarrow dim'less $T = T_{\rm dim}/\lambda^{1/3} \equiv 1/r_{\beta}$
- Low $T \ll 1$ and large number of colors $N \longrightarrow$ classical supergravity (SUGRA) Large N suppresses string quantum (g_s) corrections Low temperatures (large λ) suppress α' corrections (string size $\propto \sqrt{\alpha'}$)
- Numerical state of the art: gauge groups SU(16)-SU(32) with L up to 32
- Investigate dual black hole internal energy \leftrightarrow SYM bosonic action
- Fits to SUGRA prediction $[E/N^2 = a_0T^{2.8} + a_1T^{4.6} + a_2T^{5.8} + \dots$, with $a_0 = 7.41]$ reproduce $a_0 = 7.4(5)$ and predict unknown $a_1 = -10.0(4)$, $a_2 = 5.8(5)$
- Aside: Gauge field may not matter (arXiv:1802.00428, arXiv:1802.02985)

Two-dimensional $\mathcal{N} = (8, 8)$ SYM

• Naive dimensional reduction of twisted 4d $\mathcal{N} = 4$ SYM

$$S = \frac{N}{4\lambda} \mathcal{Q} \int d^4 x \, \operatorname{Tr} \left[\chi_{ab} \mathcal{F}_{ab} + \eta [\overline{\mathcal{D}}_a, \mathcal{D}_a] - \frac{1}{2} \eta d + \epsilon_{abcde} \chi_{de} \overline{\mathcal{D}}_c \chi_{ab} \right]$$

 $a, b = 1, \cdots, 5$ now include 'flavor' and three A_2^{\star} basis vectors

• Space-time is torus with $r_{\beta} = 1/t = \beta \sqrt{\lambda}$, $r_L = L\sqrt{\lambda}$ and aspect ratio $\alpha = r_L/r_{\beta}$ \longrightarrow more complicated phase diagram

- Solid expectations for phase transitions at both high and low temperatures Same sort of transition in each limit, with different dependence on r_β vs. r_L (All phases still thermally deconfined ↔ dual stringy black holes)
- High temperatures: Bosonic quantum mechanics transition Wilson line order parameter $W_L = \frac{1}{N} \left\langle \left| \operatorname{Tr} \left[\mathcal{P}e^{i \oint_L A} \right] \right| \right\rangle$ around spatial circle $W_L = 0$ at large r_L ('spatial confinement') $\longrightarrow W_L \neq 0$ ('deconf.') at small r_L (Order of transition debated:

first-order vs. strong second-order plus nearby Gross-Witten-Wadia)

• Low temperatures: Large-N classical SUGRA transition

Large- r_L homogeneous D1 'black strings' with horizon $\mathbb{R}\times S^7$

 \longrightarrow small- r_L D0 black holes with horizon S^8 localized on spatial circle (Radial direction U and time fill out 10 dimensions in total)

Type IIB SUGRA has winding mode instability at small $r_L \leq c_{GL} r_{\beta}^2$,

related to Type IIA classical Gregory–Laflamme transition by T duality

• Non-orthogonal basis vectors of triangular lattice

 \longrightarrow skewed tori, "generalized" thermal ensemble

• **Restricted** $SL(2,\mathbb{Z})$ modular transformations describe same torus geometry

despite sometimes changing skewed \longrightarrow rectangular

$$\begin{pmatrix} \vec{L'} \\ \vec{\beta'} \end{pmatrix} = M \cdot \begin{pmatrix} \vec{L} \\ \vec{\beta} \end{pmatrix} \qquad M = \begin{pmatrix} a & 2n \\ c & 2m-1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

with $n, m, c \in \mathbb{Z} \longrightarrow a \in 2\mathbb{Z} - 1$

 Numerical results: Horizon ↔ distribution of Wilson line eigenvalue phases In D0 / spatially deconfined phase, distribution more localized as N increases, E/(N²λ) ∝ t^{3.2} from leading-order SUGRA In D1 / spatially confined phase, distribution more uniform as N increases,

 $E/(N^2\lambda) \propto t^3$ from leading-order SUGRA

• Large-N continuum extrapolations remain to be done in this case

Deformed quantum mechanics

• Dim'l reduction of 10d plane-wave background preserving all 16 supersymmetries $S=S_0-\delta S$

$$\delta S = \frac{N}{2\lambda} \int dt \, \operatorname{Tr}\left[\frac{\mu^2}{3} (X^i)^2 + \frac{\mu^2}{6} (X^a)^2 + \frac{\mu}{24} \Psi^{\alpha} \epsilon_{ijk} \left(\gamma^i \gamma^j \gamma^k\right)_{\alpha\beta} \Psi^{\beta} + i \frac{2\mu}{3} \epsilon_{ijk} X^i X^j X^k\right]$$

with $i, j, k = 1, 2, 3$ and $a = 4, \cdots, 9$

 \longrightarrow dim'ful $\mu \neq 0$ breaks SO(9) R symmetry to SO(3)×SO(6)

- Deformation lifts moduli space \longrightarrow discrete set of vacua Can also regulate low-t instability, though this may need small $\mu \sim 1/N$
- Now have non-trivial phase diagram in plane of T/μ vs. dim'less $g = \lambda/\mu^3$ Can consider strong coupling at both large and small T/μ
- Phase diagram / transition "qualitatively similar" to 2d $\mathcal{N} = (8, 8)$, [arXiv:1411.5541] although lose thermal deconfinement \longrightarrow energy scales $\propto N^0$ rather than N^2 (no dual black holes?)
- First-order Hagedorn transition at g = 0Hawking–Page-like transition as $g \to \infty$

Higher dimensions, d = 3 and 4

- Empirically, larger d allow low-temperature stability with smaller N
 Compare 16 ≤ N for quantum mechanics vs. 6 ≤ N ≤ 16 for 2d N = (8,8) SYM
 Also expect small corrections ∝ 1/N² for adjoint fermions in 4d gauge theories
- Interpret as trading d.o.f. between space-time volume and internal large N?
- Work in progress: 3d 16-supercharge SYM in uniform D2 phase, $4 \le N \le 6$ Preliminary consistency with leading SUGRA prediction $E/(N^2\lambda^3) \propto t^{10/3}$
- For the future: "3/4 problem" in four-dimensional $\mathcal{N} = 4$ (16-supercharge) SYM Perturbative energy $1 \times cN^2T^4$ for small $\lambda \to 0$ Holographic energy $\frac{3}{4} \times cN^2T^4$ for large $1 \ll \lambda \ll N$



Figure 1: Lattice results for the dual black hole internal energy of D0 brane quantum mechanics, from arXiv:1606.04951. This work considers gauge groups SU(N) with $16 \leq N \leq 32$ on lattices with up to L = 32 sites. These two plots fix the dimensionless temperature $T = T_{\rm dim}/\lambda^{1/3} = 0.5$ and show both individual and combined extrapolations to the limits $N^2 \to \infty$ and $L \to \infty$. The latter is the continuum limit in which the lattice UV cutoff is removed while the large-N limit suppresses string quantum (g_s) corrections in the dual gravitational calculation.



Figure 2: Extrapolated lattice results for the dual black hole internal energy of D0 brane quantum mechanics, from arXiv:1606.04951, using combined $N^2 \to \infty$ and $L \to \infty$ extapolations like the one shown in Fig. 1 for T = 0.5. The results are consistently below the leading-order SUGRA prediction (solid black line), but can be fit to expressions including subleading corrections (three colored curves with error bands). These fits correctly reproduce the leading $a_0 = 7.41$ predicted by SUGRA, and provide lattice predictions for the unknown coefficients a_i for $i \ge 1$. The two colored curves without error bands are results from earlier studies with smaller N and L.



Figure 3: Schematic phase diagram for two-dimensional $\mathcal{N} = (8, 8)$ SYM on an $r_{\beta} \times r_L$ torus, from arXiv:1709.07025, showing the two limits where first-order transitions are expected. At high temperatures (small $r_{\beta} = 1/t$) the system reduces to a simple one-dimensional bosonic quantum mechanics (BQM) with a first-order deconfinement transition at small r_L . A similar first-order deconfinement transition is predicted by holography at low temperatures (in the large-N limit), with the large- r_L homogeneous black string (D1) phase becoming unstable and collapsing to a localized black hole (D0) phase as r_L decreases.



Figure 4: The complex modular parameters $\tau = \alpha \gamma + i\alpha \sqrt{1 - \gamma^2}$ for skewed tori with skewing parameter $\gamma = -1/2$ and different aspect ratios α given by the labels on the red points, from arXiv:1709.07025. When τ falls outside the shaded fundamental domain, a restricted SL(2, Z) modular transformation gives the equivalent τ' in the fundamental domain (blue points). This reveals a few cases ($\alpha = 1/2$, 4 and 8) for which the fundamental representation of the torus geometry is rectangular, $\operatorname{Re}(\tau') = 0$.



Figure 5: Numerical lattice results for two-dimensional $\mathcal{N} = (8,8)$ SYM, from arXiv:1709.07025. Top: Representative signals for the 'spatial deconfinement' transition in the spatial Wilson line (left) and its susceptibility (right), for fixed aspect ratio $\alpha = L/N_t = 4$. In the deconfined small- r_L phase at small $r_{\beta} = r_L/\alpha = 1/t$, the Wilson line is large and independent of N, while it vanishes in the large-N limit at large r, with a clear peak in the susceptibility at the transition between these two phases. Center: The resulting predictions for phase transitions for various α , compared to the expected asymptotic behavior from Fig. 3. There is good agreement at high temperatures, and reasonable consistency at lower temperatures. Bottom: The dual black hole internal energy, as in Fig. 2 but without large-N or continuum extrapolations. The $\alpha = 1/2$ data are consistent with the leading gravitational prediction $E/(N^2\lambda) \propto t^{3/2}$ for the D0 phase (left), while those for $\alpha = 2$ are consistent with $E/(N^2\lambda) \propto t^3$ for the D1 phase (right).



Figure 6: The distributions of Wilson line eigenvalue phases for two-dimensional $\mathcal{N} = (8,8)$ SYM (from arXiv:1709.07025) indicate which side of the transition we are on for a given temperature $t = 1/r_{\beta}$ and aspect ratio $\alpha = L/N_t$. For $\alpha = 1/2$ and $t \approx 0.46$ (left), the distributions become more localized as N increases, corresponding to the D0 phase (and the highest \otimes in the central plot of Fig. 5). For $\alpha = 2$ and $t \approx 0.3$, the distributions become more uniform as N increases, corresponding to the D1 phase (far beyond the right edge of the central plot in Fig. 5).



Figure 7: Preliminary lattice results for the dual black hole internal energy of three-dimensional 16supercharge SYM in the uniform D2 phase, as in the bottom plots of Fig. 5. Despite the smaller $N \leq 6$ the results are quite close to the leading gravitational prediction $E/(N^2\lambda^3) \propto t^{10/3}$.