

# Maximally supersymmetric Yang–Mills on the lattice

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Continuum and Lattice Approaches to the Infrared Behavior  
of Conformal and Quasi-Conformal Gauge Theories

Simons Center for Geometry and Physics, Stony Brook

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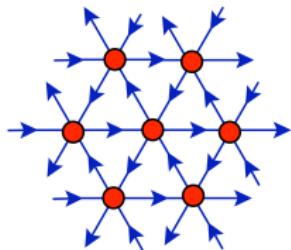
[arXiv:1505.03135](https://arxiv.org/abs/1505.03135)   [arXiv:1611.06561](https://arxiv.org/abs/1611.06561)   [arXiv:1709.07025](https://arxiv.org/abs/1709.07025)

& more to come with Simon Catterall, Raghav Jha and Toby Wiseman

# Overview and plan

## Central idea

Preserve (some) susy in discrete space-time  
to make lattice investigations practical

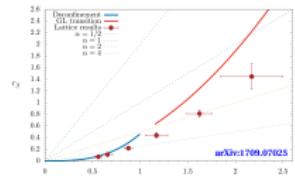
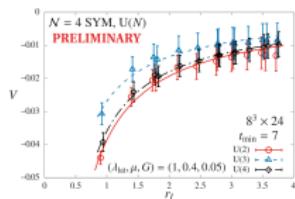
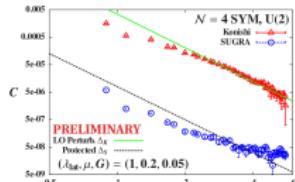


## Lattice supersymmetry

## Lattice $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM)

Selected results as time permits

- Anomalous dimension of Konishi operator
- Static potential Coulomb coefficient
- Dimensionally reduced (2d) thermodynamics



## Prospects and future directions

# Motivation: Why lattice supersymmetry

Dualities, holography, confinement, conformality, BSM, ...

Lattice promises non-perturbative insights from first principles

Many potential lattice susy applications...

- Compute Wilson loops, spectrum, scaling dimensions, etc., going beyond perturbation theory, holography, bootstrap
- New non-perturbative tests of conjectured dualities
- Predict low-energy constants from dynamical susy breaking
- Validate or refine holographic models for QCD phase diagram, condensed matter systems, etc.

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... relatively little exploration

# Obstruction: Why not lattice supersymmetry

Supersymmetry extends 4d Poincaré symmetry

by  $4\mathcal{N}$  spinor generators  $Q_\alpha^I$  and  $\bar{Q}_{\dot{\alpha}}^I$  ( $I = 1, \dots, \mathcal{N}$ )

Super-Poincaré algebra includes  $\{Q_\alpha^I, \bar{Q}_{\dot{\alpha}}^J\} = 2\delta^{IJ}\sigma_{\alpha\dot{\alpha}}^\mu P_\mu$

→ infinitesimal translations that don't exist in discrete space-time

## Consequences for lattice calculations

Explicitly broken supersymmetry  $\implies$  relevant susy-violating operators

Typically many such operators, especially with scalar fields

Fine-tuning to recover supersymmetric continuum limit

generally not practical in numerical lattice calculations

# Solution: Exact supersymmetry on the lattice

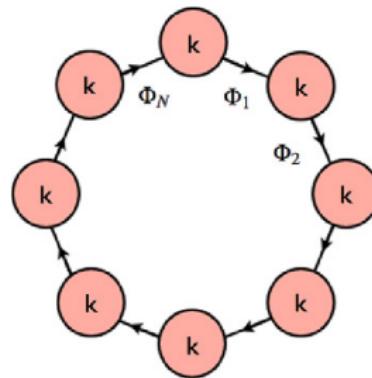
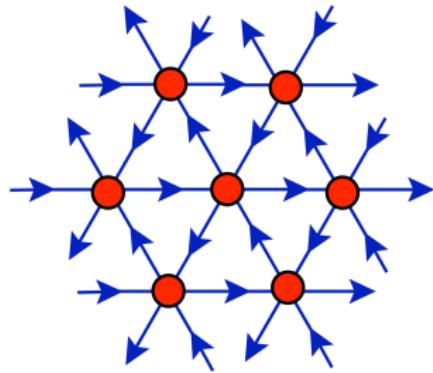
If  $2^d$  supersymmetries in  $d$  dimensions,  
can preserve susy sub-algebra at non-zero lattice spacing

⇒ Correct continuum limit with little or no fine tuning

Equivalent constructions

arXiv:0903.4881

from ‘topological’ twisting and dimensional deconstruction



In 4d pick out maximally supersymmetric Yang–Mills ( $\mathcal{N} = 4$  SYM)

# $\mathcal{N} = 4$ SYM is particularly interesting

Widely used to develop continuum QFT tools & techniques,  
from scattering amplitudes to holography

Arguably simplest non-trivial 4d field theory

$SU(N)$  gauge theory with four fermions  $\psi^I$  and six scalars  $\phi^{IJ}$ ,  
all massless and in adjoint rep.

Action consists of kinetic, Yukawa and four-scalar terms  
with coefficients related by symmetries

Maximal 16 supersymmetries  $Q_\alpha^I$  and  $\bar{Q}_{\dot{\alpha}}^I$  ( $I = 1, \dots, 4$ )  
transforming under global  $SU(4) \sim SO(6)$  R symmetry

Conformal:  $\beta$  function is zero for any 't Hooft coupling  $\lambda = g^2 N$

# Topological twisting for $\mathcal{N} = 4$ SYM

Intuitive picture — expand  $4 \times 4$  matrix of supersymmetries

$$\begin{pmatrix} Q_\alpha^1 & Q_\alpha^2 & Q_\alpha^3 & Q_\alpha^4 \\ \bar{Q}_{\dot{\alpha}}^1 & \bar{Q}_{\dot{\alpha}}^2 & \bar{Q}_{\dot{\alpha}}^3 & \bar{Q}_{\dot{\alpha}}^4 \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \bar{\mathcal{Q}} \gamma_5$$
$$\rightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$

with  $a, b = 1, \dots, 5$

Kähler–Dirac multiplet of ‘twisted’ supersymmetries  $\mathcal{Q}$   
transforming with integer spin under ‘twisted rotation group’

$$\mathrm{SO}(4)_{tw} \equiv \mathrm{diag} \left[ \mathrm{SO}(4)_{\mathrm{euc}} \otimes \mathrm{SO}(4)_R \right] \quad \mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$$

Change of variables  $\rightarrow$  closed subalgebra  $\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$   
that can be **exactly preserved on the lattice**

# Susy subalgebra from twisted $\mathcal{N} = 4$ SYM

Fields &  $\mathcal{Q}$ s transform with integer spin under  $SO(4)_{tw}$  — no spinors

$Q_\alpha$  and  $\bar{Q}_{\dot{\alpha}}$   $\rightarrow \mathcal{Q}$ ,  $\mathcal{Q}_a$  and  $\mathcal{Q}_{ab}$

$\psi$  and  $\bar{\psi}$   $\rightarrow \eta$ ,  $\psi_a$  and  $\chi_{ab}$

$A_\mu$  and  $\Phi^I$   $\rightarrow$  complexified gauge field  $\mathcal{A}_a$  and  $\bar{\mathcal{A}}_a$

Complexification  $\rightarrow U(N) = SU(N) \otimes U(1)$  gauge theory

Schematically, under the twisted  $SO(d)_{tw} = \text{diag} [SO(d)_{\text{euc}} \otimes SO(d)_R]$

$A_\mu \sim \text{vector} \otimes \text{scalar} \rightarrow \text{vector}$

$\Phi^I \sim \text{scalar} \otimes \text{vector} \rightarrow \text{vector}$

Easiest to see by dimensionally reducing from 5d

$$\mathcal{A}_a = A_a + i\Phi_a \rightarrow (A_\mu, \phi) + i(\Phi_\mu, \bar{\phi})$$

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Fields &  $\mathcal{Q}$ s transform with integer spin under  $SO(4)_{tw}$  — no spinors

$Q_\alpha$  and  $\bar{Q}_{\dot{\alpha}}$   $\rightarrow \mathcal{Q}$ ,  $Q_a$  and  $Q_{ab}$

$\psi$  and  $\bar{\psi}$   $\rightarrow \eta$ ,  $\psi_a$  and  $\chi_{ab}$

$A_\mu$  and  $\Phi^I$   $\rightarrow$  complexified gauge field  $\mathcal{A}_a$  and  $\bar{\mathcal{A}}_a$

Twisted-scalar supersymmetry  $\mathcal{Q}$

correctly interchanges bosonic  $\longleftrightarrow$  fermionic d.o.f. with  $\mathcal{Q}^2 = 0$

$$\mathcal{Q} \mathcal{A}_a = \psi_a$$

$$\mathcal{Q} \psi_a = 0$$

$$\mathcal{Q} \chi_{ab} = -\bar{\mathcal{F}}_{ab}$$

$$\mathcal{Q} \bar{\mathcal{A}}_a = 0$$

$$\mathcal{Q} \eta = d$$

$$\mathcal{Q} d = 0$$



bosonic auxiliary field with e.o.m.  $d = \bar{\mathcal{D}}_a \mathcal{A}_a$

# Lattice $\mathcal{N} = 4$ SYM

Lattice theory nearly a direct transcription despite breaking  $\mathcal{Q}_a$  and  $\mathcal{Q}_{ab}$

Covariant derivatives  $\rightarrow$  finite difference operators

Complexified gauge fields  $A_a \rightarrow$  gauge links  $U_a \in \mathfrak{gl}(N, \mathbb{C})$

$$\mathcal{Q} A_a \rightarrow \mathcal{Q} U_a = \psi_a \quad \mathcal{Q} \psi_a = 0$$

$$\mathcal{Q} \chi_{ab} = -\bar{\mathcal{F}}_{ab} \quad \mathcal{Q} \bar{A}_a \rightarrow \mathcal{Q} \bar{U}_a = 0$$

$$\mathcal{Q} \eta = d \quad \mathcal{Q} d = 0$$

(geometrically  $\eta$  on sites,  $\psi_a$  on links, etc.)

Susy lattice action ( $\mathcal{Q}S = 0$ ) from  $\mathcal{Q}^2 \cdot = 0$  and **Bianchi identity**

$$S = \frac{N}{4\lambda_{\text{lat}}} \text{Tr} \left[ \mathcal{Q} \left( \chi_{ab} \mathcal{F}_{ab} + \eta \bar{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \bar{D}_c \chi_{de} \right]$$

# Five links in four dimensions $\longrightarrow A_4^*$ lattice

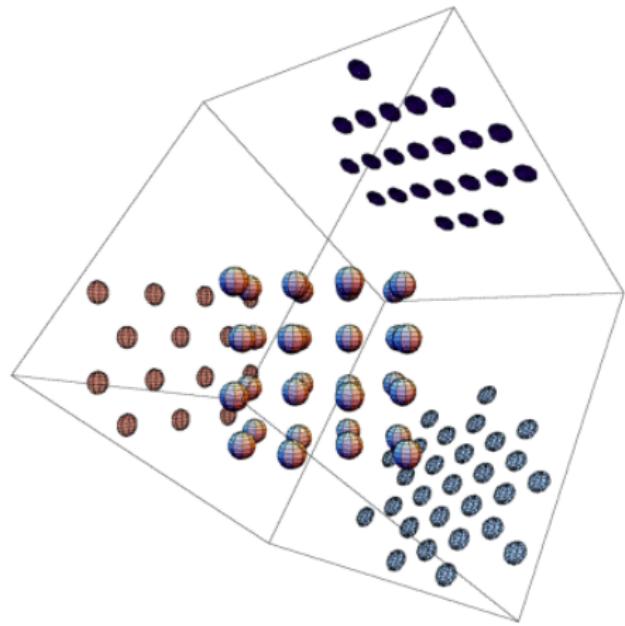
Again easiest to dimensionally reduce from 5d,

treating all five gauge links  $U_a$  symmetrically

Start with hypercubic lattice  
in 5d momentum space

**Symmetric** constraint  $\sum_a \partial_a = 0$   
projects to 4d momentum space

Result is  $A_4$  lattice  
 $\longrightarrow$  dual  $A_4^*$  lattice in real space

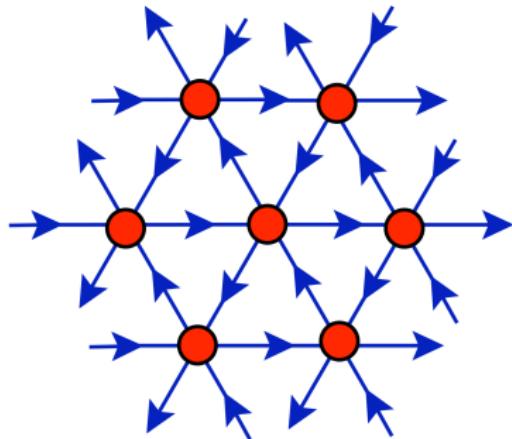


# Twisted SO(4) symmetry on the $A_4^*$ lattice

Can view  $A_4^*$  lattice  
as 4d analog of 2d triangular lattice

Basis vectors linearly dependent  
and non-orthogonal  $\rightarrow \lambda = \lambda_{\text{lat}}/\sqrt{5}$

Preserves  $S_5$  point group symmetry



$S_5$  irreps precisely match onto irreps of twisted  $\text{SO}(4)_{tw}$

$$\mathbf{5} = \mathbf{4} \oplus \mathbf{1} : \quad \psi_a \longrightarrow \psi_\mu, \quad \bar{\eta}$$

$$\mathbf{10} = \mathbf{6} \oplus \mathbf{4} : \quad \chi_{ab} \longrightarrow \chi_{\mu\nu}, \quad \bar{\psi}_\mu$$

$S_5 \longrightarrow \text{SO}(4)_{tw}$  in continuum limit restores  $\mathcal{Q}_a$  and  $\mathcal{Q}_{ab}$

# Summary of twisted $\mathcal{N} = 4$ SYM on the $A_4^*$ lattice

$U(N)$  gauge invariance +  $\mathcal{Q}$  +  $S_5$  lattice symmetries  
→ several significant analytic results

Moduli space preserved to all orders of lattice perturbation theory  
→ no scalar potential induced by radiative corrections

$\beta$  function vanishes at one loop in lattice perturbation theory

Real-space RG blocking transformations preserving  $\mathcal{Q}$  and  $S_5$   
→ no new terms in long-distance effective action

Only one log. tuning to recover continuum  $\mathcal{Q}_a$  and  $\mathcal{Q}_{ab}$

**Not quite suitable for numerical calculations**

Exact zero modes and flat directions must be regulated,  
especially important in  $U(1)$  sector

# Regulating SU( $N$ ) flat directions

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[ \mathcal{Q} \left( \chi_{ab} \mathcal{F}_{ab} + \eta \bar{\mathcal{D}}_a \mathcal{U}_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \bar{\mathcal{D}}_c \chi_{de} + \mu^2 V \right]$$

Scalar potential  $V = \sum_a \left( \frac{1}{N} \text{Tr} [\mathcal{U}_a \bar{\mathcal{U}}_a] - 1 \right)^2$  lifts SU( $N$ ) flat directions  
and ensures  $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$  in continuum limit

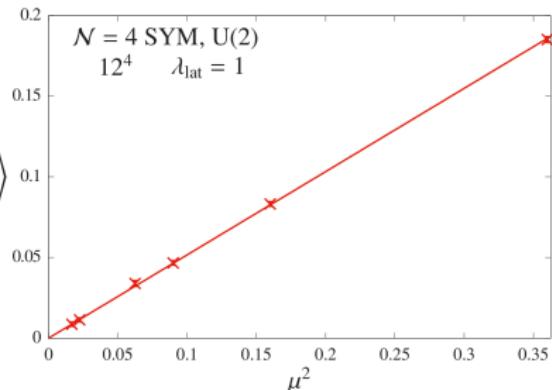
Softly breaks  $\mathcal{Q}$  — all susy violations  $\propto \mu^2 \rightarrow 0$  in continuum limit

Ward identity violations,  $\langle \mathcal{QO} \rangle \neq 0$ ,  
show  $\mathcal{Q}$  breaking and restoration

$$\left\langle \frac{|\mathcal{QO}|}{\sqrt{D^2 + F^2}} \right\rangle$$

Here considering

$$\mathcal{Q} [\eta \mathcal{U}_a \bar{\mathcal{U}}_a] = d \mathcal{U}_a \bar{\mathcal{U}}_a - \eta \psi_a \bar{\mathcal{U}}_a$$



# Full $\mathcal{N} = 4$ SYM lattice action

arXiv:1505.03135

$$\mathcal{S} = \frac{N}{4\lambda_{\text{lat}}} \left[ \mathcal{Q} \left( \chi_{ab} \mathcal{F}_{ab} + \downarrow - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \bar{\mathcal{D}}_c \chi_{de} + \mu^2 V \right]$$

$$\eta \left\{ \bar{\mathcal{D}}_a \mathcal{U}_a + G \sum_{a < b} [\det \mathcal{P}_{ab} - 1] \mathbb{I}_N \right\}$$

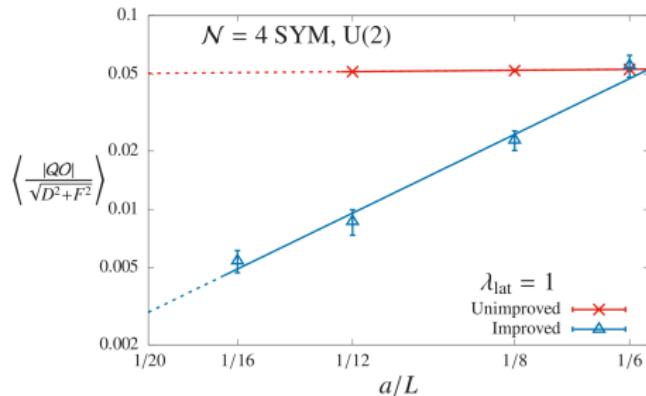
Modify e.o.m. for  $d$  to constrain **plaquette determinant**

→ lifts U(1) zero mode & flat directions without susy breaking

Much better than adding

another soft  $\mathcal{Q}$ -breaking term

$O(a)$  improvement,  $\langle \mathcal{QO} \rangle \propto (a/L)^2$ ,  
since  $\mathcal{Q}$  forbids all dim-5 operators



# Advertisement: Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$S_{\text{imp}} = S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \quad (3.10)$$

$$\begin{aligned} S'_{\text{exact}} &= \frac{N}{2\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ -\bar{\mathcal{F}}_{ab}(n) \mathcal{F}_{ab}(n) - \chi_{ab}(n) \mathcal{D}_{[a}^{(+)} \psi_{b]}(n) - \eta(n) \bar{\mathcal{D}}_a^{(-)} \psi_a(n) \right. \\ &\quad \left. + \frac{1}{2} \left( \bar{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) + G \sum_{a \neq b} (\det \mathcal{P}_{ab}(n) - 1) \mathbb{I}_N \right)^2 \right] - S_{\text{det}} \end{aligned}$$

$$S_{\text{det}} = \frac{N}{2\lambda_{\text{lat}}} G \sum_n \text{Tr} [\eta(n)] \sum_{a \neq b} [\det \mathcal{P}_{ab}(n)] \text{Tr} [\mathcal{U}_b^{-1}(n) \psi_b(n) + \mathcal{U}_a^{-1}(n + \hat{\mu}_b) \psi_a(n + \hat{\mu}_b)]$$

$$S_{\text{closed}} = -\frac{N}{8\lambda_{\text{lat}}} \sum_n \text{Tr} [\epsilon_{abcde} \chi_{dc}(n + \hat{\mu}_a + \hat{\mu}_b + \hat{\mu}_c) \bar{\mathcal{D}}_c^{(-)} \chi_{ab}(n)],$$

$$S'_{\text{soft}} = \frac{N}{2\lambda_{\text{lat}}} \mu^2 \sum_n \sum_a \left( \frac{1}{N} \text{Tr} [\mathcal{U}_a(n) \bar{\mathcal{U}}_a(n)] - 1 \right)^2$$

The full  $\mathcal{N} = 4$  SYM lattice action is somewhat complicated  
( $\gtrsim 100$  inter-node data transfers in the fermion operator)

To reduce barriers to entry our parallel code is publicly developed at  
[github.com/daschaich/susy](https://github.com/daschaich/susy)

Evolved from MILC lattice QCD code, presented in [arXiv:1410.6971](https://arxiv.org/abs/1410.6971)

## Application: Konishi operator scaling dimension

Conformality  $\rightarrow$  spectrum of scaling dimensions  $\Delta(\lambda)$   
govern power-law decays of correlation functions

Konishi is simplest conformal primary operator

$$\mathcal{O}_K(x) = \sum_I \text{Tr} [\Phi^I(x) \Phi^I(x)] \quad C_K(r) \equiv \mathcal{O}_K(x+r) \mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

Predictions for Konishi scaling dimension  $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$

- From weak-coupling perturbation theory,  
related to strong coupling by  $\frac{4\pi N}{\lambda} \longleftrightarrow \frac{\lambda}{4\pi N}$  S duality
- From holography for  $N \rightarrow \infty$  and  $\lambda \rightarrow \infty$  with  $\lambda \ll N$
- Upper bounds from conformal bootstrap

Only lattice gauge theory can access nonperturbative  $\lambda$  at moderate  $N$

# Konishi operator on the lattice

Scalar fields  $\varphi(n)$  from polar decomposition of complexified links

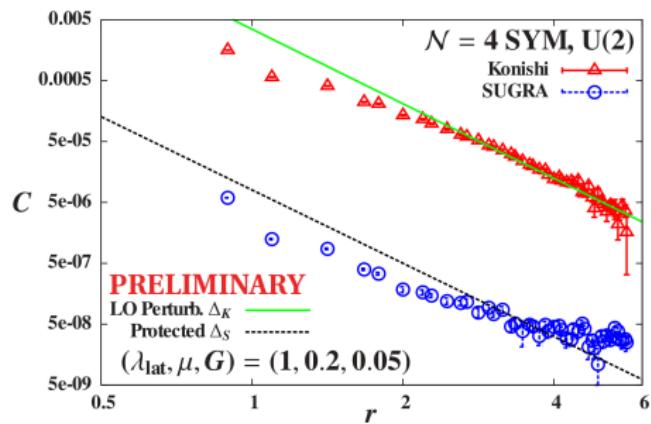
$$\mathcal{U}_a(n) \longrightarrow e^{\varphi_a(n)} U_a(n) \quad \mathcal{O}_K^{\text{lat}}(n) = \sum_a \text{Tr} [\varphi_a(n) \varphi_a(n)] - \text{vev}$$

Also looking at ‘SUGRA’ (20’)

$\mathcal{O}_S \sim \varphi_a \varphi_b$  with protected  $\Delta_S = 2$

Challenging systematics from  
directly fitting power-law decay

Better lattice tools to find  $\Delta$ :  
Finite-size scaling  
Monte Carlo RG



Need lattice RG blocking transformation to carry out MCRG...

# Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve  $\mathcal{Q}$  and  $S_5$  symmetries  $\longleftrightarrow$  geometric structure

Simple transformation constructed in [arXiv:1408.7067](#)

$$\begin{aligned}\mathcal{U}'_a(n') &= \xi \mathcal{U}_a(n) \mathcal{U}_a(n + \hat{\mu}_a) & \eta'(n') &= \eta(n) \\ \psi'_a(n') &= \xi [\psi_a(n) \mathcal{U}_a(n + \hat{\mu}_a) + \mathcal{U}_a(n) \psi_a(n + \hat{\mu}_a)] & \text{etc.}\end{aligned}$$

Doubles lattice spacing  $a \rightarrow a' = 2a$ , with  $\xi$  a tunable rescaling factor

Scalar fields from polar decomposition  $\mathcal{U}(n) = e^{\varphi(n)} U(n)$   
are shifted,  $\varphi \rightarrow \varphi + \log \xi$ , since blocked  $U$  must remain unitary

$\mathcal{Q}$ -preserving RG blocking needed

to show only one log. tuning to recover continuum  $\mathcal{Q}_a$  and  $\mathcal{Q}_{ab}$

## Scaling dimensions from MCRG stability matrix

System as (infinite) sum of operators  $H = \sum_i c_i \mathcal{O}_i$

Couplings  $c_i$  flow under RG blocking  $R_b$

$n$ -times-blocked system  $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Fixed point defined by  $H^* = R_b H^*$  with couplings  $c_i^*$

Linear expansion around fixed point defines **stability matrix**  $T_{ij}^*$

$$c_i^{(n)} - c_i^* = \sum_k \left. \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \right|_{H^*} (c_k^{(n-1)} - c_k^*) \equiv \sum_j T_{ik}^* (c_k^{(n-1)} - c_k^*)$$

Correlators of  $\mathcal{O}_i, \mathcal{O}_k \rightarrow$  elements of stability matrix [Swendsen, 1979]

Eigenvalues of  $T_{ik}^*$   $\rightarrow$  scaling dimensions of corresponding operators

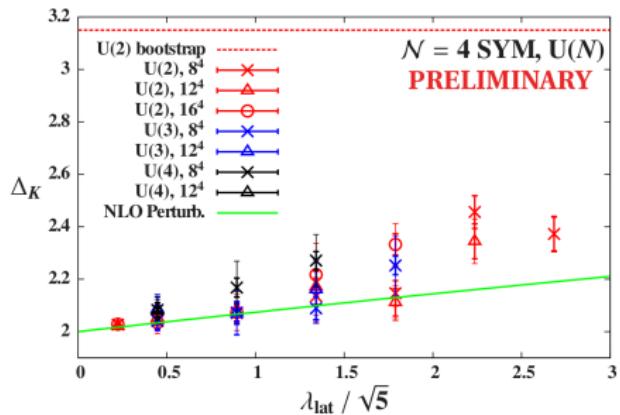
# Preliminary $\Delta_K$ results from Monte Carlo RG

MCRG stability matrix

includes both  $\mathcal{O}_K^{\text{lat}}$  and  $\mathcal{O}_S^{\text{lat}}$

Impose protected  $\Delta_S = 2$

Systematic uncertainties from  
different amounts of smearing



Complication: Twisted  $\text{SO}(4)_{tw}$  involves only  $\text{SO}(4)_R \subset \text{SO}(6)_R$

⇒ Lattice Konishi operator mixes with  $\text{SO}(4)_R$ -singlet part  
of the  $\text{SO}(6)_R$ -nonsinglet SUGRA operator

Working on variational analyses to disentangle operators

# Application: Static potential

Static potential  $V(r)$  from  $r \times T$  Wilson loops     $W(r, T) \propto e^{-V(r)T}$

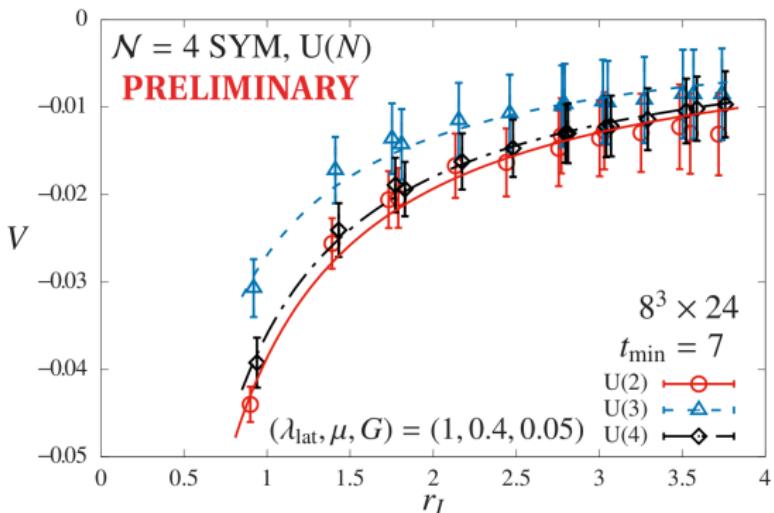
Fit  $V(r)$  to Coulombic or confining form

$$V(r) = A - C/r$$

$$V(r) = A - C/r + \sigma r$$

$C$  is Coulomb coefficient

$\sigma$  is string tension



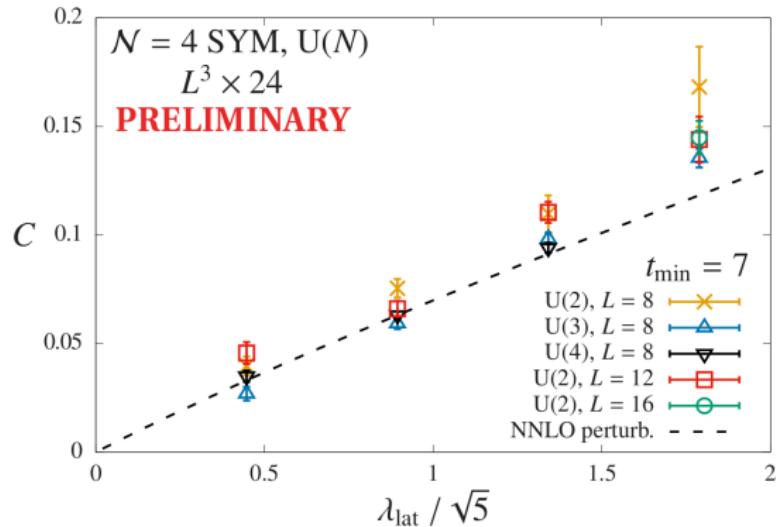
$V(r)$  is Coulombic at all  $\lambda$  (fits to confining form produce vanishing  $\sigma$ )

Tree-level improved analysis reduces discretization artifacts

# Coupling dependence of Coulomb coefficient

Continuum perturbation theory predicts  $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

Holography predicts  $C(\lambda) \propto \sqrt{\lambda}$  for  $N \rightarrow \infty$  and  $\lambda \rightarrow \infty$  with  $\lambda \ll N$



Results consistent with perturbation theory

for these relatively weak couplings  $\lambda_{\text{lat}} \leq 4$

# Application: Thermodynamics on a 2-torus

Improve arXiv:1008.4964 with new parallel code

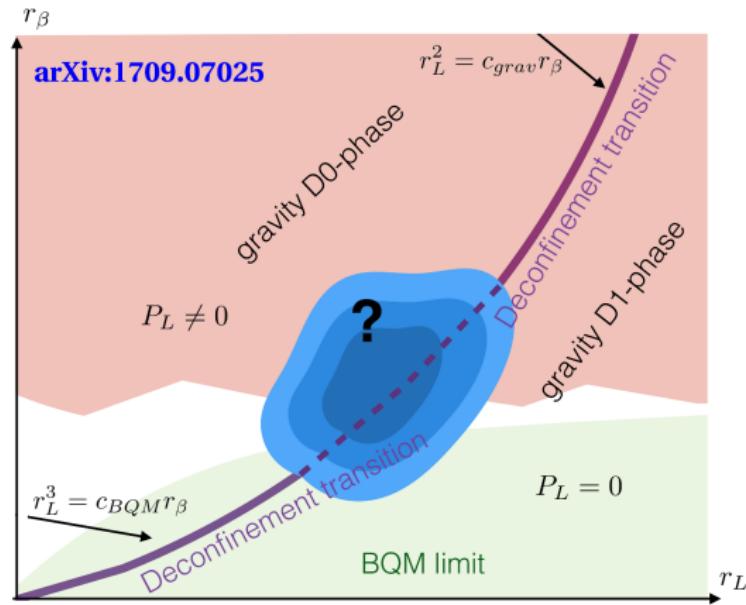
Dimensionally reduce to 2d  $\mathcal{N} = (8, 8)$  SYM with four scalar  $Q$ ,  
study low temperatures  $t = 1/r_\beta \longleftrightarrow$  black holes in dual supergravity

For decreasing  $r_L$  **at large  $N$**

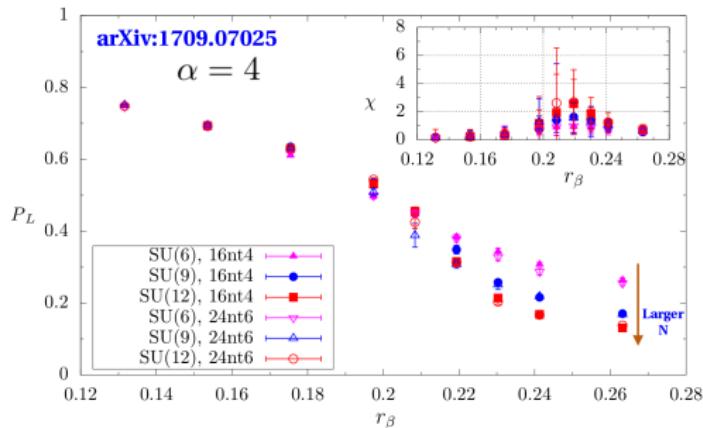
homogeneous black string (D1)  
 $\longrightarrow$  localized black hole (D0)



“spatial deconfinement”  
signalled by Wilson line  $P_L$



# $\mathcal{N} = (8, 8)$ SYM lattice phase diagram results

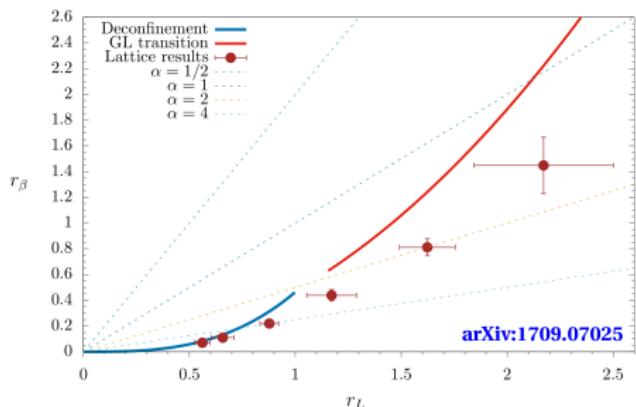


Fix aspect ratio  $\alpha = r_L/r_\beta$ ,  
 scan in  $r_\beta = r_L/\alpha = \beta\sqrt{\lambda}$

Transition at peak  
 of Wilson line susceptibility  $\chi$

Lower-temperature transitions  
 at smaller  $\alpha < 1 \rightarrow$  larger errors

Results consistent with holography  
 and high-temp. bosonic QM



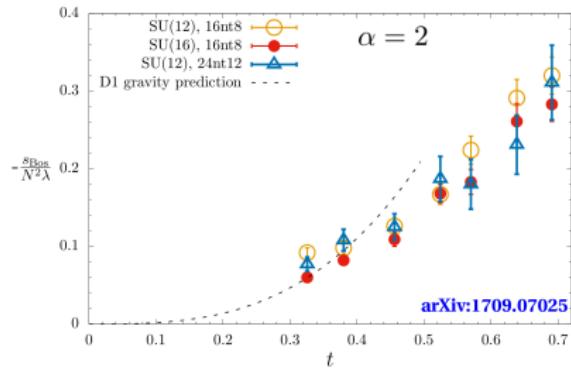
# Dual black hole thermodynamics

Holography predicts bosonic action for corresponding dual black holes

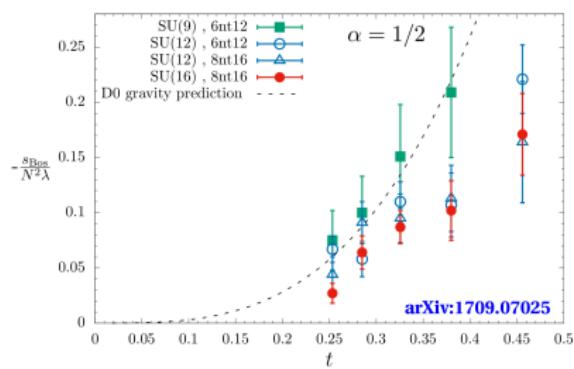
$$S_{\text{Bos}} \propto t^3 \text{ for large-}r_L \text{ D1 phase}$$

$$S_{\text{Bos}} \propto t^{3.2} \text{ for small-}r_L \text{ D0 phase}$$

Lattice results consistent with holography for sufficiently low  $t \lesssim 0.4$



arXiv:1709.07025



Need larger  $N > 16$  to avoid instabilities at lower temperatures

# Recapitulation and outlook

## Significant progress in lattice supersymmetry

- Lattice promises non-perturbative insights from first principles
  - Lattice  $\mathcal{N} = 4$  SYM is practical thanks to exact  $\mathcal{Q}$  susy
  - Public code to reduce barriers to entry
- 
- Progress toward conformal scaling dimension of Konishi operator
  - Static potential Coulomb coefficient  $C(\lambda)$  at weak coupling
  - 2d  $\mathcal{N} = (8, 8)$  SYM thermodynamics consistent with holography

## Many more directions are being — or can be — pursued

- Understanding the (absence of a) sign problem
- Systems with less supersymmetry, in lower dimensions, including matter fields, exhibiting spontaneous susy breaking, ...

## Upcoming Workshops

Numerical approaches to holography,  
quantum gravity and cosmology

21–24 May 2018

Higgs Centre for Theoretical Physics, Edinburgh

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Interdisciplinary approach  
to QCD-like composite dark matter

1–5 October 2018

ECT\* Trento

# Thank you!

# Thank you!

## Collaborators

Simon Catterall, Raghav Jha, Toby Wiseman

also Georg Bergner, Poul Damgaard, Joel Giedt, Anosh Joseph

## Funding and computing resources



# Supplement: Lattice superQCD in 2d & 3d

Add fundamental matter multiplets without breaking  $\mathcal{Q}^2 = 0$

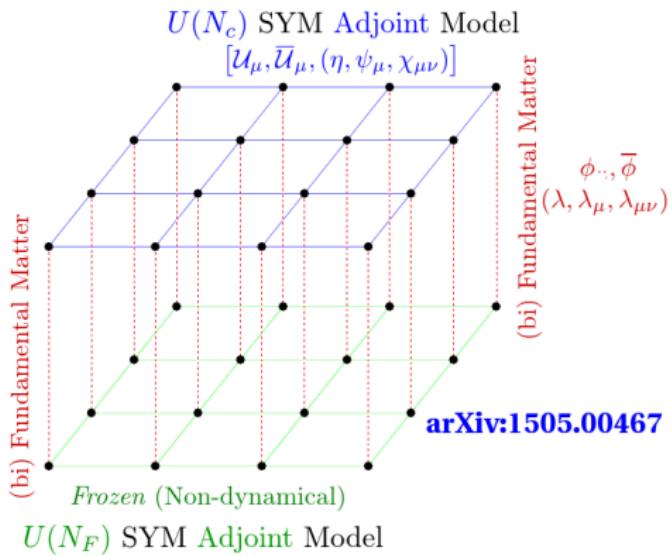
Proposed by Matsuura [[arXiv:0805.4491](#)] and Sugino [[arXiv:0807.2683](#)],  
first numerical study by Catterall & Veernala [[arXiv:1505.00467](#)]

2-slice lattice SYM  
with  $U(N) \times U(F)$  gauge group

Adj. fields on each slice

Bi-fundamental in between

Set  $U(F)$  gauge coupling to zero  
 $\rightarrow U(N)$  in  $d - 1$  dims.  
with  $F$  fund. hypermultiplets



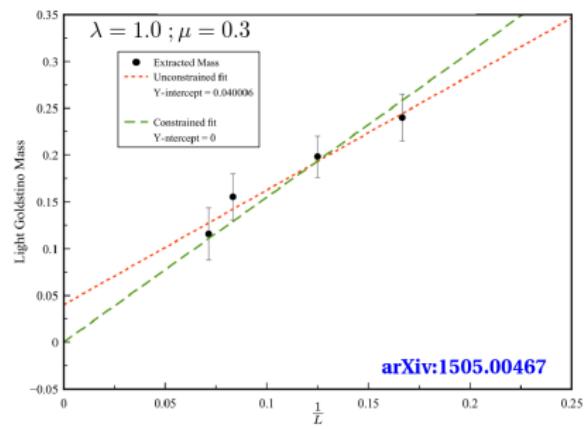
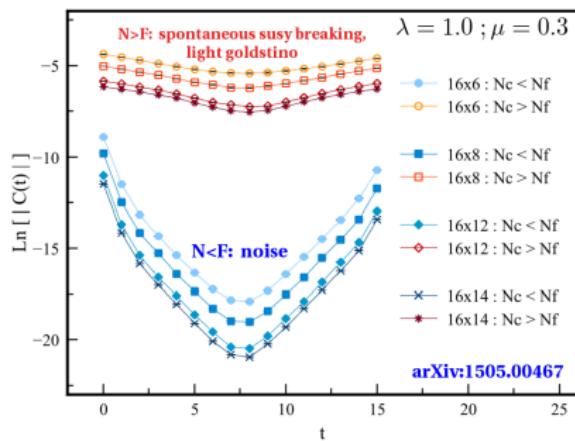
# Spontaneous supersymmetry breaking

Auxiliary field e.o.m.  $\rightarrow$  Fayet–Iliopoulos  $D$ -term potential

$$d = \bar{\mathcal{D}}_a \mathcal{U}_a + \sum_{i=1}^F \phi_i \bar{\phi}_i + r \mathbb{I}_N \quad \rightarrow \quad S_D \propto \sum_{i=1}^F \text{Tr} [\phi_i \bar{\phi}_i + r \mathbb{I}_N]^2$$

$\langle Q\eta \rangle = \langle d \rangle \neq 0 \implies \langle 0 | H | 0 \rangle > 0$  (spontaneous susy breaking)

$\rightarrow N \times N$  conditions vs.  $N \times F$  degrees of freedom



## Supplement: Potential sign problem

Observables:  $\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}] [d\bar{\mathcal{U}}] \mathcal{O} e^{-S_B[\mathcal{U}, \bar{\mathcal{U}}]} \text{pf } \mathcal{D}[\mathcal{U}, \bar{\mathcal{U}}]$

Pfaffian can be complex for lattice  $\mathcal{N} = 4$  SYM,  $\text{pf } \mathcal{D} = |\text{pf } \mathcal{D}| e^{i\alpha}$

Complicates interpretation of  $\{e^{-S_B} \text{pf } \mathcal{D}\}$  as Boltzmann weight

RHMC uses **phase quenching**,  $\text{pf } \mathcal{D} \rightarrow |\text{pf } \mathcal{D}|$ , needs reweighting

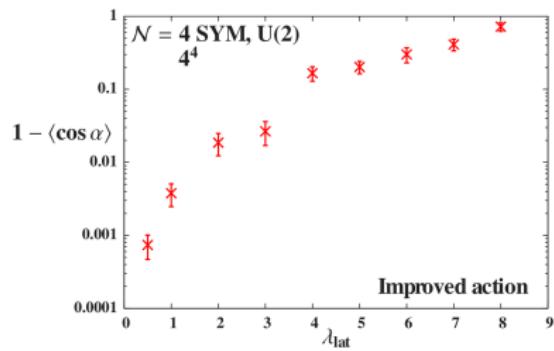
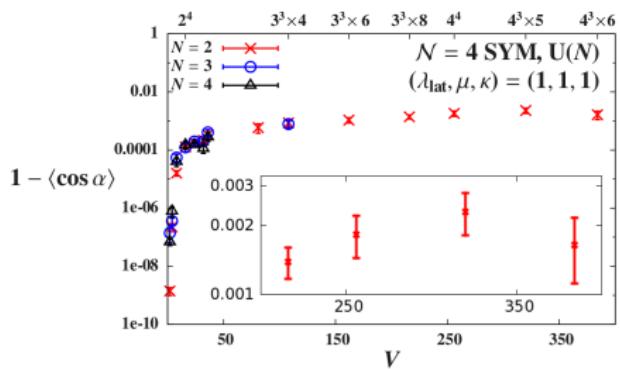
$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}} \quad \text{with} \quad \langle \mathcal{O} e^{i\alpha} \rangle_{pq} = \frac{1}{\mathcal{Z}_{pq}} \int [d\mathcal{U}] [d\bar{\mathcal{U}}] \mathcal{O} e^{i\alpha} e^{-S_B} |\text{pf } \mathcal{D}|$$

⇒ Monitor  $\langle e^{i\alpha} \rangle_{pq}$  as function of volume, coupling,  $N$

# Pfaffian phase dependence on volume and coupling

**Left:**  $1 - \langle \cos(\alpha) \rangle_{pq} \ll 1$  independent of volume and  $N$  at  $\lambda_{\text{lat}} = 1$

**Right:** Larger  $\lambda_{\text{lat}} \geq 4 \rightarrow$  much larger phase fluctuations



**To do:** Analyze more volumes and  $N$  with improved action

Extremely expensive  $\mathcal{O}(n^3)$  computation

$\sim 50 \text{ hours} \times 16 \text{ cores}$  for single U(2)  $4^4$  measurement

# Two puzzles posed by the sign problem

Periodic temporal boundary conditions for the fermions

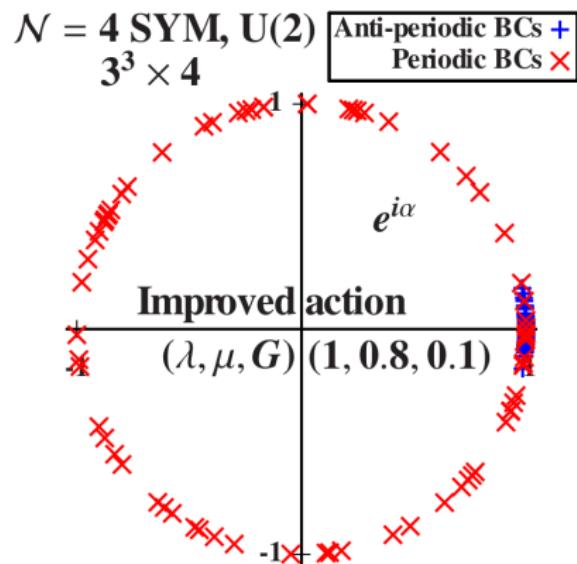
→ obvious sign problem,  $\langle e^{i\alpha} \rangle_{pq} \approx 0$

Anti-periodic BCs →  $e^{i\alpha} \approx 1$ , phase reweighting negligible

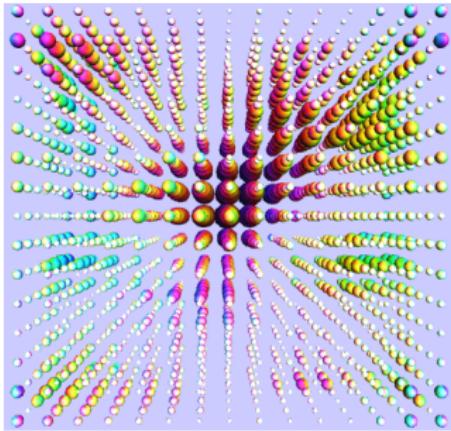
Why such sensitivity to the BCs?

Other  $pq$  observables  
are nearly identical  
for these two ensembles

Why doesn't the sign problem  
affect other observables?



# Backup: Essence of numerical lattice calculations



(Image credit: Claudio Rebbi)

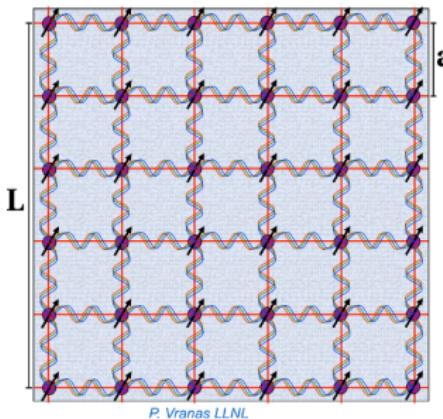
Evaluate observables from functional integral  
via importance sampling Monte Carlo

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U \ \mathcal{O}(U) \ e^{-S[U]}$$
$$\longrightarrow \frac{1}{N} \sum_{i=1}^N \mathcal{O}(U_i) \text{ with uncert. } \propto \sqrt{\frac{1}{N}}$$

$U$  are field configurations in discretized euclidean space-time,  
sampled with probability  $\propto e^{-S}$

$S[U]$  is lattice action,  
should be real and positive  $\longrightarrow \frac{1}{\mathcal{Z}} e^{-S}$  as probability distribution

## Backup: More features of lattice calculations



Spacing “ $a$ ” between lattice sites  
→ UV cutoff scale  $1/a$

Removing cutoff:  $a \rightarrow 0$  (with  $L/a \rightarrow \infty$ )

Lattice cutoff preserves hypercubic subgroup  
→ restore Poincaré in continuum limit

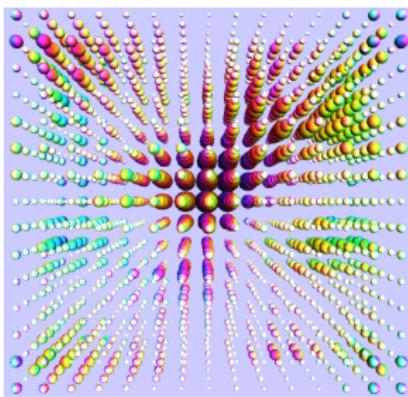
Lattice action  $S$  defined by bare lagrangian at the UV cutoff  $1/a$

After generating and saving ensembles  $\{U_n\}$  distributed  $\propto e^{-S}$   
often quick and easy to measure many observables  $\langle \mathcal{O} \rangle$

Changing the action (generally) requires generating new ensembles

# Backup: Hybrid Monte Carlo (HMC) algorithm

Goal: Sample field configurations  $U$  with probability  $\frac{1}{Z} e^{-S[U]}$



(Image credit: Claudio Rebbi)

HMC is Markov process based on  
Metropolis–Rosenbluth–Teller

Fermions  $\rightarrow$  extensive action computation

$\Rightarrow$  Global updates  
using fictitious molecular dynamics

- ① Introduce fictitious “MD time”  $\tau$   
and stochastic canonical momenta for fields
- ② Inexact MD evolution along trajectory in  $\tau \rightarrow$  new configuration
- ③ Accept/reject test on MD discretization error

## Backup: Failure of Leibnitz rule in discrete space-time

$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu = 2i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu$  is problematic

→ try  $\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2i\sigma_{\alpha\dot{\alpha}}^\mu \nabla_\mu$  for a discrete translation

$$\nabla_\mu \phi(x) = \frac{1}{a} [\phi(x + a\hat{\mu}) - \phi(x)] = \partial_\mu \phi(x) + \frac{a}{2} \partial_\mu^2 \phi(x) + \mathcal{O}(a^2)$$

Essential difference between  $\partial_\mu$  and  $\nabla_\mu$  on the lattice,  $a > 0$

$$\begin{aligned}\nabla_\mu [\phi(x)\eta(x)] &= a^{-1} [\phi(x + a\hat{\mu})\eta(x + a\hat{\mu}) - \phi(x)\eta(x)] \\ &= [\nabla_\mu \phi(x)] \eta(x) + \phi(x) \nabla_\mu \eta(x) + a [\nabla_\mu \phi(x)] \nabla_\mu \eta(x)\end{aligned}$$

Only recover Leibnitz rule  $\partial_\mu(fg) = (\partial_\mu f)g + f\partial_\mu g$  when  $a \rightarrow 0$

⇒ “Discrete supersymmetry” breaks down on the lattice

(Dondi & Nicolai, “Lattice Supersymmetry”, 1977)

## Backup: Twisting $\longleftrightarrow$ Kähler–Dirac fermions

Kähler–Dirac representation related to spinor  $Q_\alpha^I$ ,  $\bar{Q}_{\dot{\alpha}}^I$  by

$$\begin{pmatrix} Q_\alpha^1 & Q_\alpha^2 & Q_\alpha^3 & Q_\alpha^4 \\ \bar{Q}_{\dot{\alpha}}^1 & \bar{Q}_{\dot{\alpha}}^2 & \bar{Q}_{\dot{\alpha}}^3 & \bar{Q}_{\dot{\alpha}}^4 \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \bar{\mathcal{Q}} \gamma_5$$
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$

with  $a, b = 1, \dots, 5$

The  $4 \times 4$  matrix involves R symmetry transformations along each row,  
(euclidean) Lorentz transformations along each column

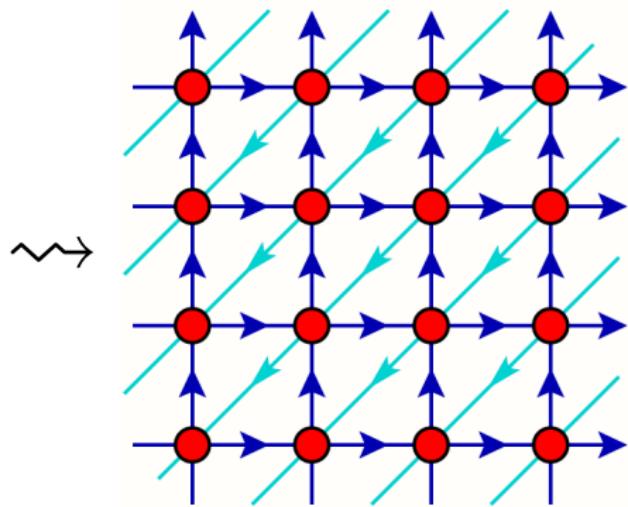
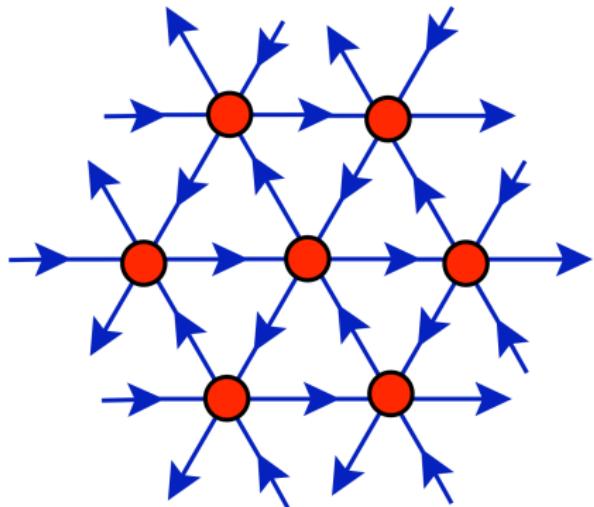
$\implies$  Kähler–Dirac components transform under “twisted rotation group”

$$SO(4)_{tw} \equiv \text{diag} \left[ SO(4)_{\text{euc}} \otimes SO(4)_R \right]$$

$\uparrow$   
only  $SO(4)_R \subset SO(6)_R$

## Backup: Hypercubic representation of $A_4^*$ lattice

In the code it is very convenient to represent the  $A_4^*$  lattice  
as a hypercube plus one backwards diagonal link



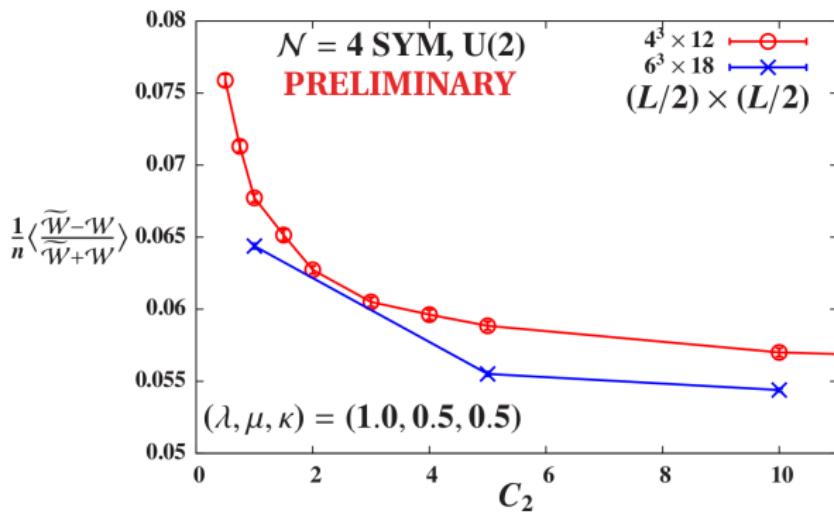
## Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

$\mathcal{Q}_a$  and  $\mathcal{Q}_{ab}$  from restoration of R symmetry (motivation for  $A_4^*$  lattice)

Modified Wilson loops test R symmetries at non-zero lattice spacing

Parameter  $c_2$  may need logarithmic tuning in continuum limit

Results from [arXiv:1411.0166](https://arxiv.org/abs/1411.0166) to be revisited with improved action



## Backup: More on flat directions

Complexified links  $\rightarrow U(N) = SU(N) \otimes U(1)$  gauge invariance

Supersymmetry transformation  $\mathcal{Q} \mathcal{U}_a = \psi_a$

$\Rightarrow$  links must be in algebra with continuum limit  $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$

Flat directions in  $SU(N)$  sector are physical,

those in  $U(1)$  sector decouple only in continuum limit

Both must be regulated in calculations  $\rightarrow$  two deformations

Scalar potential  $\propto \mu^2 \sum_a (\text{Tr} [\mathcal{U}_a \bar{\mathcal{U}}_a] - N)^2$  for  $SU(N)$  sector

Plaquette determinant  $\sim G \sum_{a < b} (\det \mathcal{P}_{ab} - 1)$  for  $U(1)$  sector

Scalar potential **softly** breaks  $\mathcal{Q}$  supersymmetry

susy-violating operators vanish as  $\mu^2 \rightarrow 0$

Plaquette determinant can be made  $\mathcal{Q}$ -invariant  $\rightarrow$  improved action

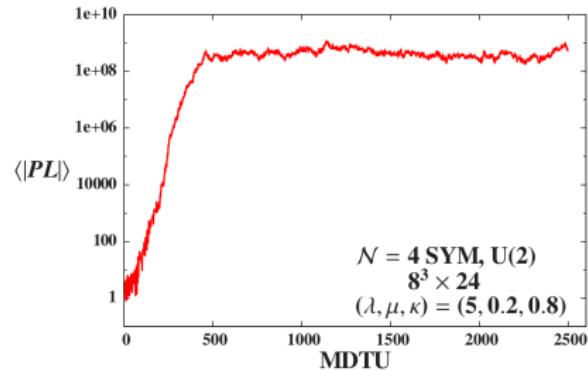
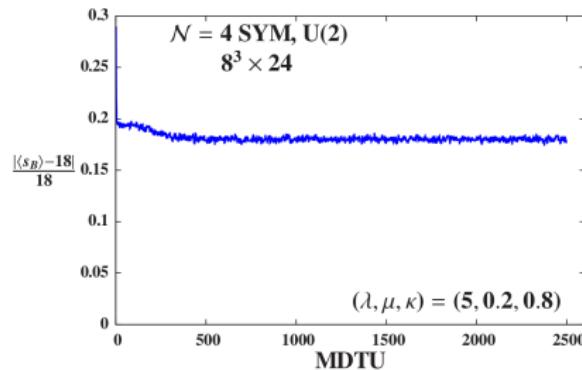
# Backup: Problem with $SU(N)$ flat directions

$\mu^2/\lambda_{\text{lat}}$  too small  $\rightarrow \mathcal{U}_a$  can move far from continuum form  $\mathbb{I}_N + \mathcal{A}_a$

Example:  $\mu = 0.2$  and  $\lambda_{\text{lat}} = 5$  on  $8^3 \times 24$  volume

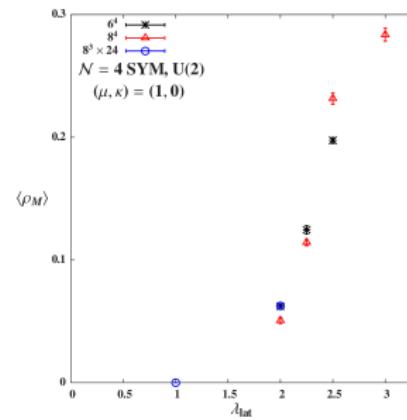
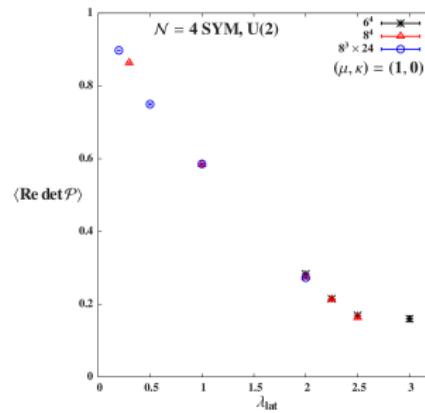
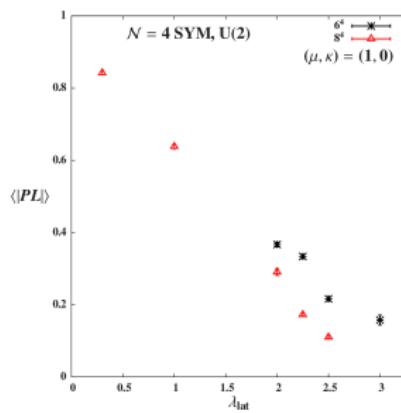
**Left:** Bosonic action stable  $\sim 18\%$  off its supersymmetric value

**Right:** Complexified Polyakov ('Maldacena') loop wanders off to  $\sim 10^9$



# Backup: Problem with U(1) flat directions

Monopole condensation  $\rightarrow$  confined lattice phase  
not present in continuum  $\mathcal{N} = 4$  SYM



Around the same  $\lambda_{\text{lat}} \approx 2\dots$

**Left:** Polyakov loop falls towards zero

**Center:** Plaquette determinant falls towards zero

**Right:** Density of U(1) monopole world lines becomes non-zero

# Backup: More on soft supersymmetry breaking

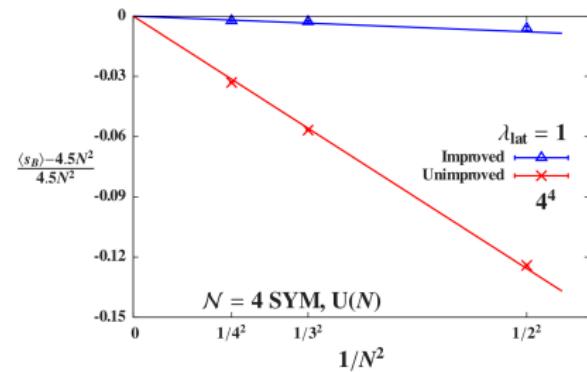
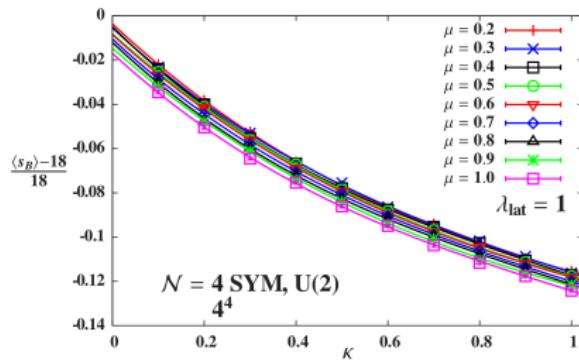
Until 2015  $(\det \mathcal{P} - 1)$  was another soft susy-breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_a \left( \frac{1}{N} \text{Tr} [\mathcal{U}_a \bar{\mathcal{U}}_a] - 1 \right)^2 + \kappa \sum_{a < b} |\det \mathcal{P}_{ab} - 1|^2$$

Much larger  $\mathcal{Q}$ -breaking effects than scalar potential

**Left:**  $\mathcal{Q}$  Ward identity from bosonic action  $\langle s_B \rangle = 9N^2/2$

**Right:** Soft susy breaking suppressed  $\propto 1/N^2$



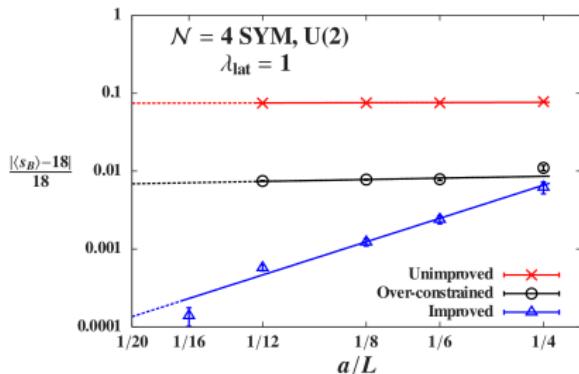
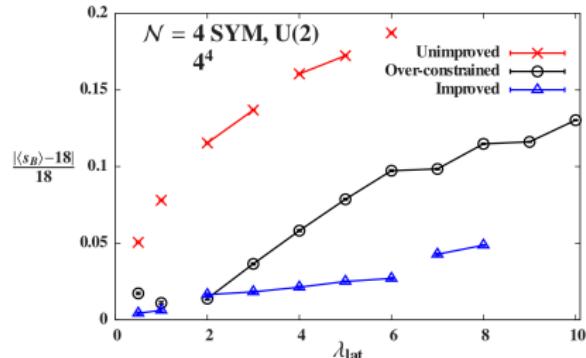
# Backup: Supersymmetric moduli space modification

arXiv:1505.03135 introduces method to impose  $\mathcal{Q}$ -invariant constraints

Modify auxiliary field equations of motion  $\rightarrow$  moduli space

$$d(n) = \overline{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) \quad \longrightarrow \quad d(n) = \overline{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

Including both plaquette determinant and scalar potential in  $\mathcal{O}(n)$   
over-constraints system  $\rightarrow$  sub-optimal Ward identity violations



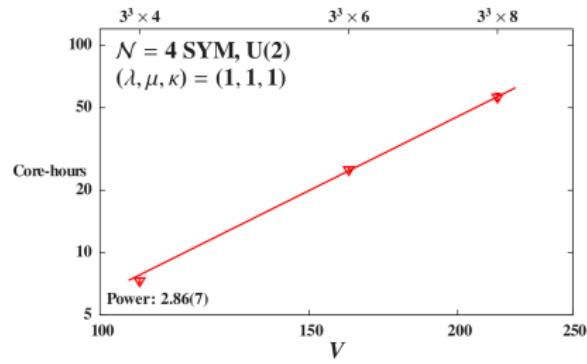
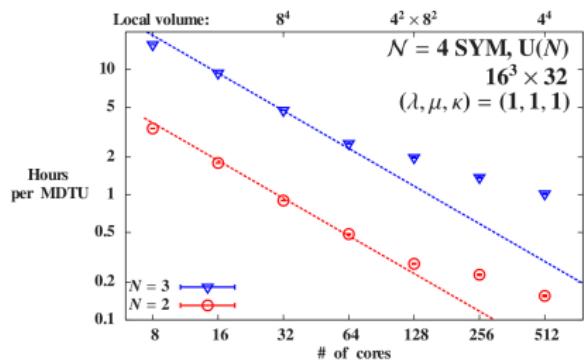
# Backup: Code performance—weak and strong scaling

Results from [arXiv:1410.6971](#) to be revisited with improved action

**Left:** Strong scaling for U(2) and U(3)  $16^3 \times 32$  RHMC

**Right:** Weak scaling for  $\mathcal{O}(n^3)$  pfaffian calculation (fixed local volume)

$n \equiv 16N^2 V$  is number of fermion degrees of freedom



Dashed lines are optimal scaling

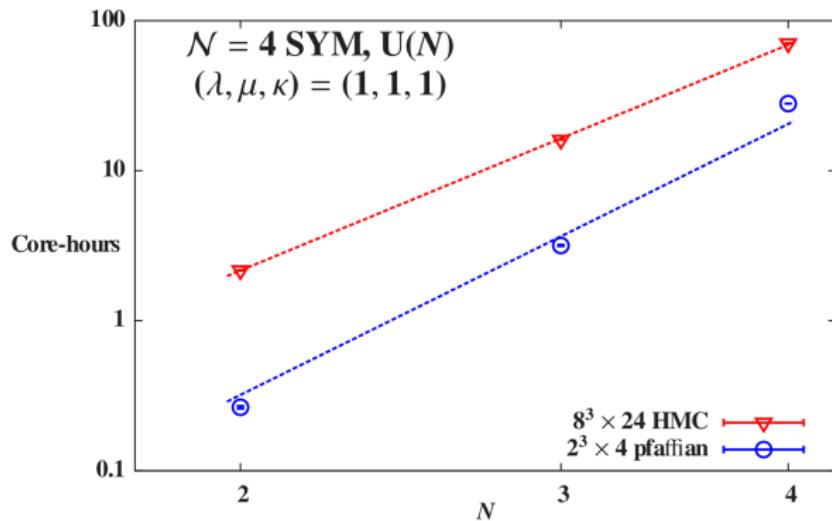
Solid line is power-law fit

## Backup: Numerical costs for $N = 2, 3$ and 4 colors

**Red:** Original RHMC cost scaling  $\sim N^5$  now improved to  $\sim N^{3.5}$

Plot from [arXiv:1410.6971](#) to be updated

**Blue:** Pfaffian cost scaling consistent with expected  $N^6$



# Backup: Smearing for Konishi analyses

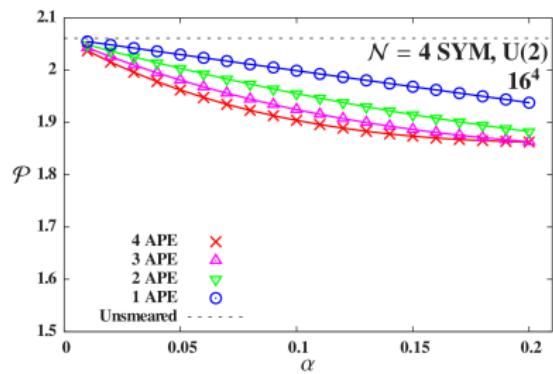
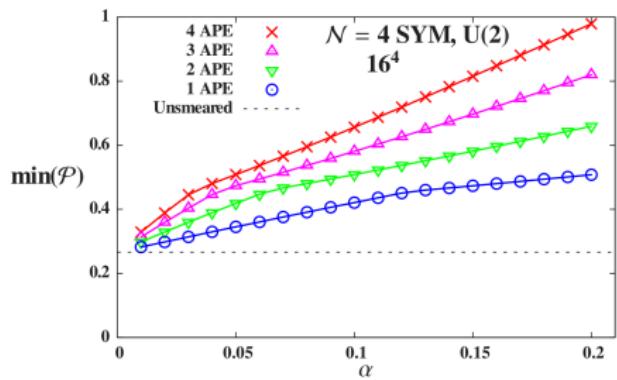
As for glueballs, smear to enlarge operator basis

APE-like smearing:  $\square \rightarrow (1 - \alpha)\square + \frac{\alpha}{8} \sum \square$ ,

staples built from unitary parts of links but no final unitarization  
(unitarized smearing — e.g. stout — doesn't affect Konishi)

Average plaquette stable upon smearing (**right**)

while minimum plaquette steadily increases (**left**)



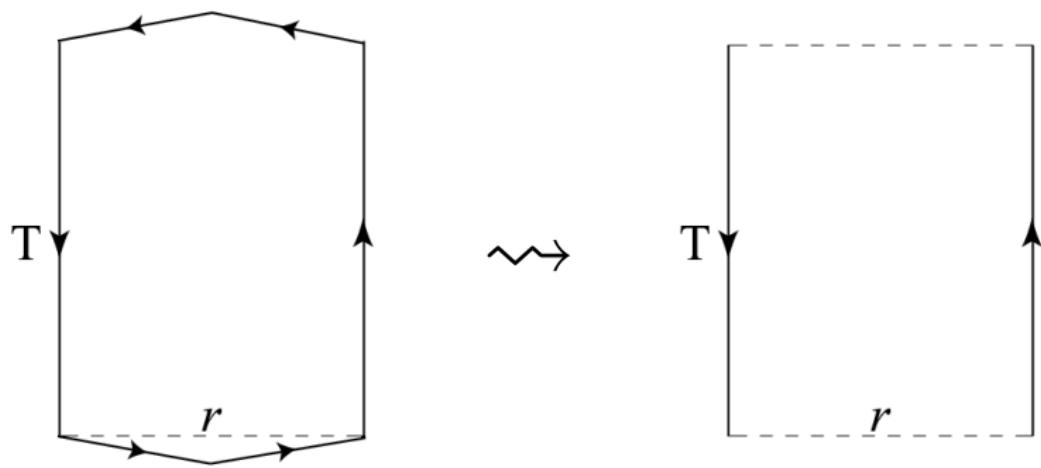
# Backup: $\mathcal{N} = 4$ SYM static potential from Wilson loops

Extract static potential  $V(r)$  from  $r \times T$  Wilson loops

$$W(r, T) \propto e^{-V(r) T}$$

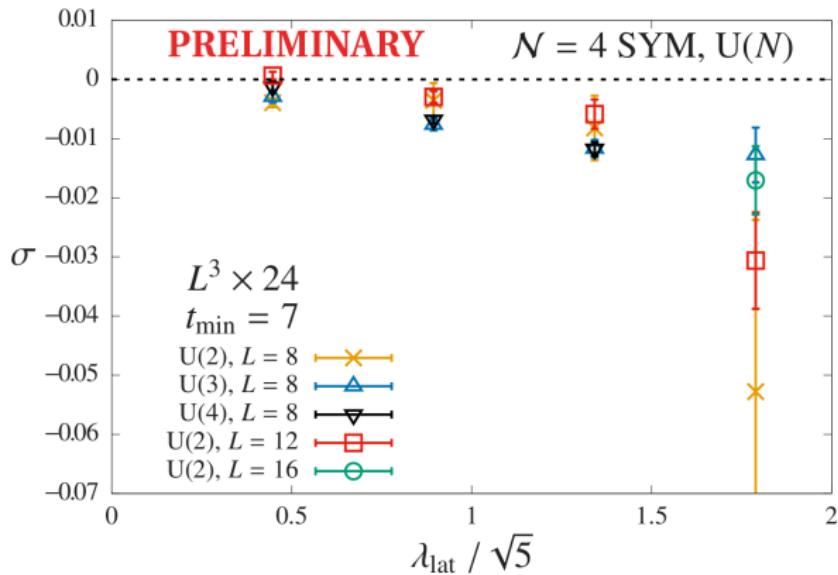
$$V(r) = A - C/r + \sigma r$$

Coulomb gauge trick from lattice QCD reduces  $A_4^*$  lattice complications



# Backup: Static potential is Coulombic at all $\lambda$

String tension  $\sigma$  from fits to confining form  $V(r) = A - C/r + \sigma r$

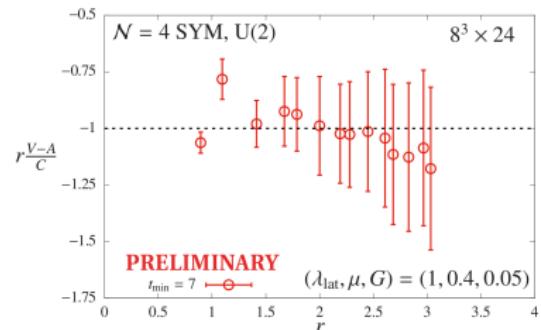
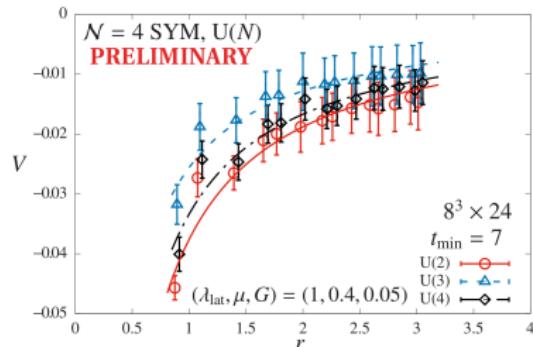


Slightly negative values flatten  $V(r_l)$  for  $r_l \lesssim L/2$

$\sigma \rightarrow 0$  as accessible range of  $r_l$  increases on larger volumes

# Backup: Tree-level improvement for static potential

Discretization artifacts visible in naive static potential analyses



Improve by applying tree-level lattice perturbation theory  
for  $\mathcal{N} = 4$  SYM bosonic propagator on  $A_4^*$  lattice:

$$V(r) \longrightarrow V(r_I) \quad \text{where} \quad \frac{1}{r_I^2} \equiv 4\pi^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\cos(ir \cdot k)}{4 \sum_{\mu=1}^4 \sin^2(k \cdot \hat{e}_{\mu}/2)}$$

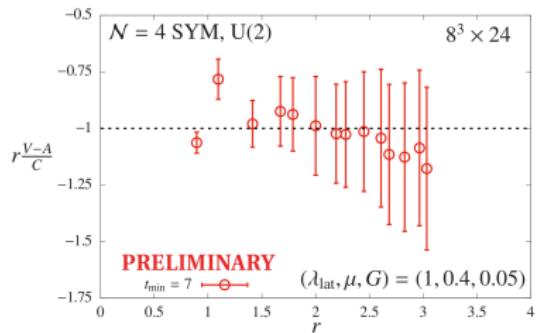
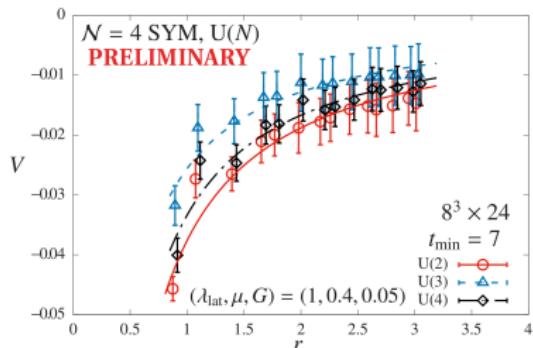
$\hat{e}_{\mu}$  are  $A_4^*$  lattice basis vectors

[arXiv:1102.1725]

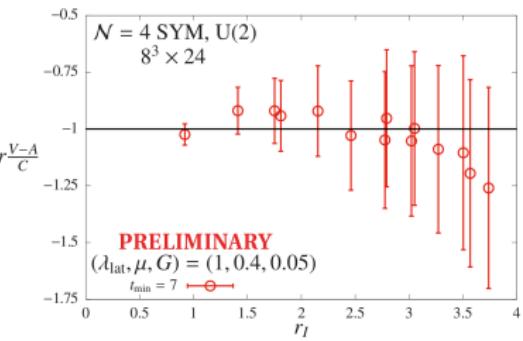
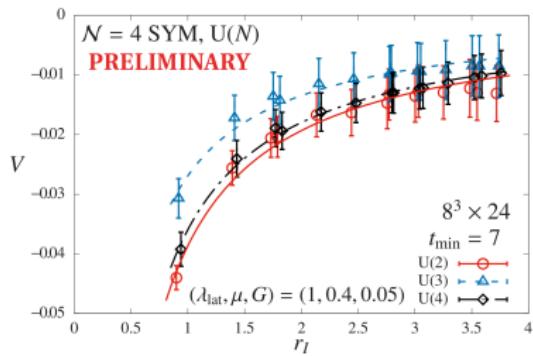
Momenta  $k = \frac{2\pi}{L} \sum_{\mu=1}^4 n_{\mu} \hat{g}_{\mu}$  depend on dual basis vectors

# Backup: Tree-level improvement for static potential

Discretization artifacts visible in naive static potential analyses



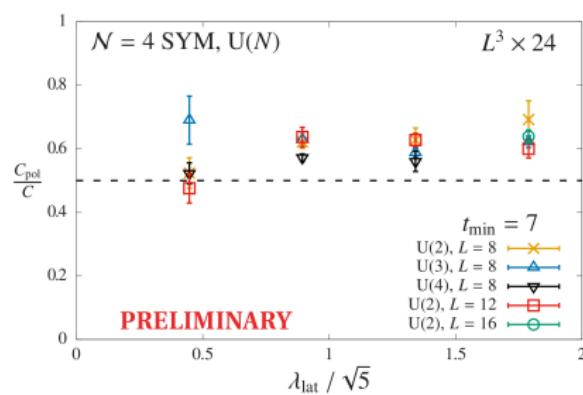
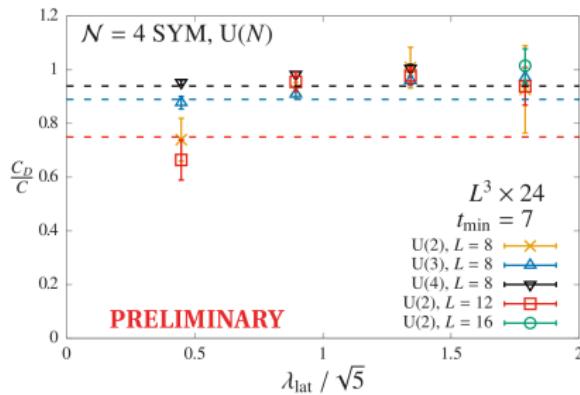
Tree-level improvement significantly reduces discretization artifacts



# Backup: More $\mathcal{N} = 4$ SYM static potential tests

**Left:** Projecting Wilson loops from  $U(N) \rightarrow SU(N) \Rightarrow$  factor of  $\frac{N^2 - 1}{N^2}$

**Right:** Unitarizing links removes scalars  $\Rightarrow$  factor of 1/2



Several ratios end up above expected values

Cause not clear — seems insensitive to lattice volume and  $\mu$

# Backup: Dimensional reduction to $\mathcal{N} = (8, 8)$ SYM

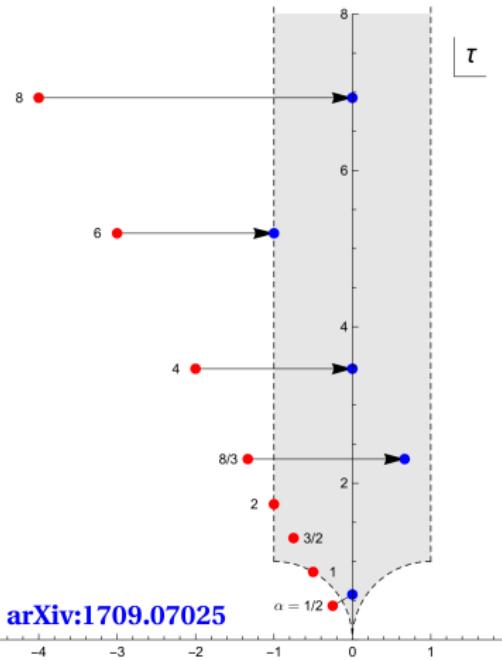
Naive for now: 4d  $\mathcal{N} = 4$  SYM code with  $N_x = N_y = 1$

$A_4^*$  lattice  $\longrightarrow A_2^*$  (triangular) lattice

$\implies$  Torus **skewed** depending on  $\alpha$

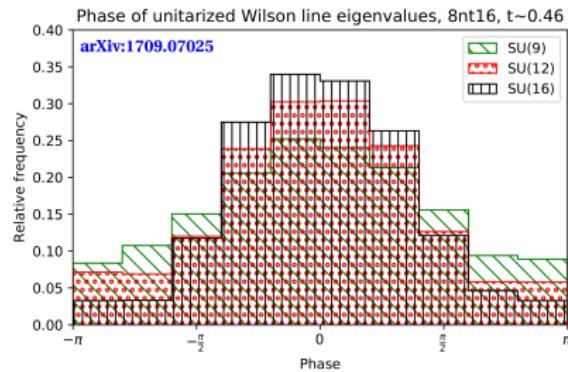
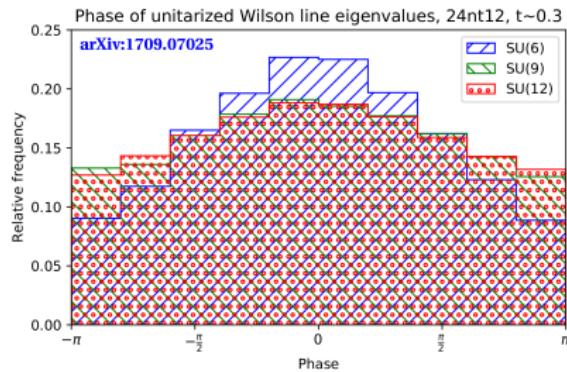
Modular trans. into fund. domain  
can make skewed torus rectangular

Also need to stabilize compactified links  
to ensure broken center symmetries



# Backup: $\mathcal{N} = (8, 8)$ SYM Wilson line eigenvalues

Check ‘spatial deconfinement’ through histograms  
of Wilson line eigenvalue phases



**Left:**  $\alpha = 2$  distributions more extended as  $N$  increases  
→ dual gravity describes homogeneous black string (D1 phase)

**Right:**  $\alpha = 1/2$  distributions more compact as  $N$  increases  
→ dual gravity describes localized black hole (D0 phase)