

Physics Out Of The Box

The impact of lattice gauge theory and advanced computing



David Schaich (Syracuse University)

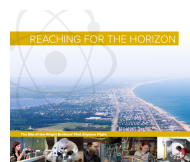
Stony Brook Nuclear Theory Seminar, 13 November 2015

dschaich@syr.edu

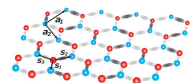
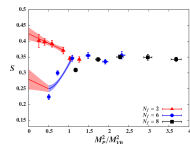
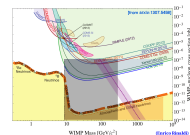
www.davidschaich.net

Some things to take away

- Computing is a tool
It doesn't do our physical understanding for us
- Lattice gauge theory is a broadly applicable tool
Used in nuclear, particle, condensed matter physics
- Computing capabilities continue to increase
→ Numerical approaches increasingly valuable



The 2015
LONG RANGE PLAN
for NUCLEAR SCIENCE

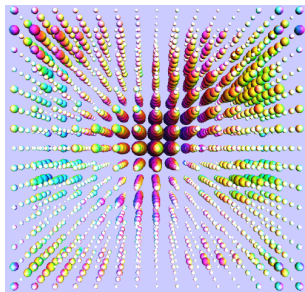


Lattice gauge theory in a nutshell

Lattice gauge theory is a fully non-perturbative and gauge-invariant regularization of quantum field theory (QFT)

Any QFT observable is formally $\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \mathcal{O}(\Phi) e^{-S[\Phi]}$

...but this is an infinite-dimensional integral



Regularize the theory by formulating it in a finite, discrete spacetime \longrightarrow **the lattice**

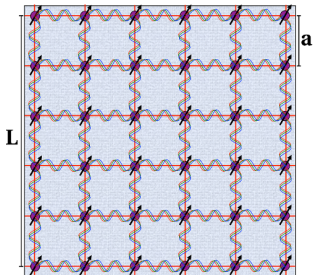
Working in Euclidean spacetime

Spacing between lattice sites (“ a ”) introduces UV cutoff scale $1/a$

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P. Vranas LLNL

Regularize the theory by formulating it
in a finite, discrete spacetime \rightarrow **the lattice**

Working in Euclidean spacetime

Spacing between lattice sites (“ a ”)
introduces UV cutoff scale $1/a$

Lattice cutoff preserves hypercubic subgroup of full Lorentz symmetry

Remove cutoff by taking continuum limit $a \rightarrow 0$ (with $L/a \rightarrow \infty$)

Numerical lattice gauge theory calculations

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \mathcal{O}(\Phi) e^{-S[\Phi]}$$

Finite-dimensional integral \implies we can compute $\langle \mathcal{O} \rangle$ numerically

Importance sampling Monte Carlo

Approximate integral with a finite ensemble of field configurations $\{\Phi_i\}$

Algorithms choose each configuration Φ_i with probability $\frac{1}{Z} e^{-S[\Phi_i]}$
to find those that make the most important contributions

Then $\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{i=1}^N \mathcal{O}(\Phi_i)$ with statistical uncertainty $\propto \sqrt{\frac{1}{N}}$

Generating ensembles $\{\Phi_i\}$ often dominates computational costs

These saved data can be reused to investigate many observables

For decades lattice gauge theory
has helped to drive advances in high-performance computing



IBM Blue Gene/Q @Livermore



USQCD cluster @Fermilab

Results to be shown are from
state-of-the-art lattice calculations

$\mathcal{O}(100\text{M})$ core-hours) invested overall

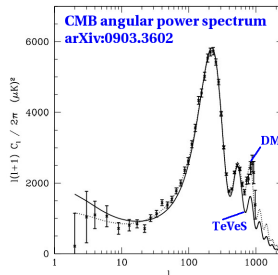
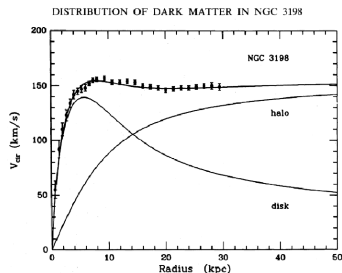
Many thanks to DOE, NSF
and computing centers!



Cray Blue Waters @NCSA

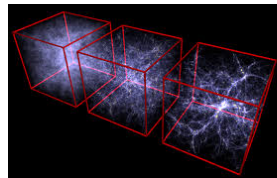
Application: Dark matter

Dark matter is 'known unknown' physics beyond the standard model

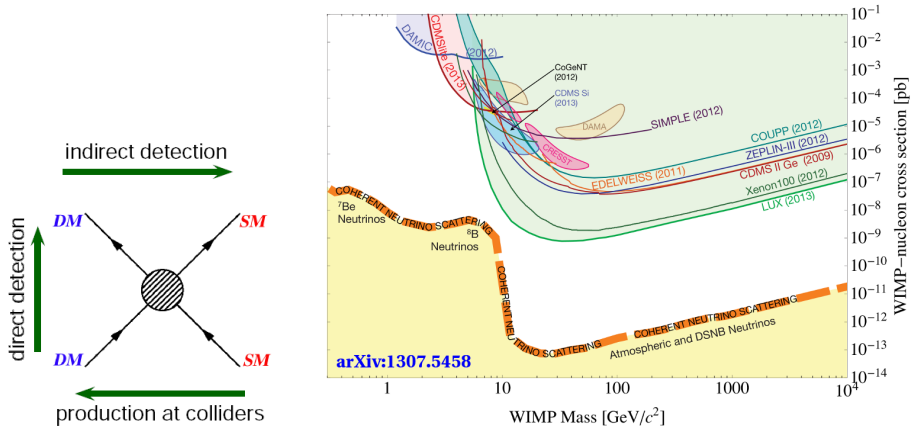


Abundant gravitational evidence on many scales

- Rotation curves of galaxies & clusters
- Gravitational lensing
- Structure formation
- Cosmological backgrounds



No clear signals in non-gravitational searches for dark matter (at colliders, in cosmic rays, and underground)



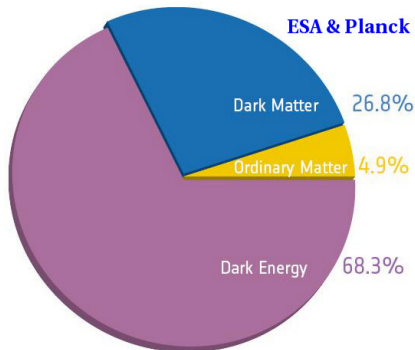
But we expect dark matter to interact with standard model fields. . .

Motivation for non-gravitational interactions

$$\frac{\Omega_{DM}}{\Omega_{SM}} \approx 5 \quad \dots \text{not } 10^5 \text{ or } 10^{-5}$$

Suggests DM–SM coupling

Common feature of thermal
and asymmetric mechanisms
for relic density generation



If relic density relies on coupling to standard model fields,
such interactions must satisfy current experimental constraints

Composite dark matter is a natural solution

Charged fermions ψ confined in **electroweak-neutral** “dark baryon”

Form factors for composite dark matter direct detection

Photon exchange via electromagnetic form factors

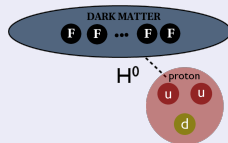
Interactions suppressed by powers of confinement scale $\Lambda \sim M_{DM}$

- **Dimension 5:** Magnetic moment $\rightarrow (\bar{\psi}\sigma^{\mu\nu}\psi) F_{\mu\nu}/\Lambda$
- **Dimension 6:** Charge radius $\rightarrow (\bar{\psi}\psi) v_\mu \partial_\nu F_{\mu\nu}/\Lambda^2$
- **Dimension 7:** Polarizability $\rightarrow (\bar{\psi}\psi) F^{\mu\nu} F_{\mu\nu}/\Lambda^3$

Higgs boson exchange via scalar form factors

Effective Higgs interaction of composite DM
needed for correct Big Bang nucleosynthesis

Higgs couples through $\langle B|m_\psi\bar{\psi}\psi|B\rangle$ (σ terms)



All form factors arise non-perturbatively \Rightarrow lattice calculations

Form factors for composite dark matter direct detection

Photon exchange via electromagnetic form factors

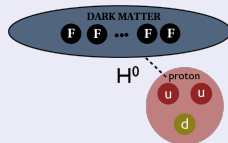
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Goal: Impose most general constraints on composite paradigm

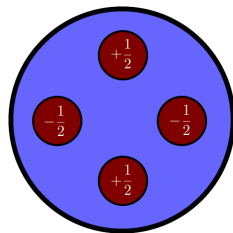
Stealth dark matter

Lattice Strong Dynamics Collaboration,

PRL **115:171803** (2015, Editors' Suggestion)

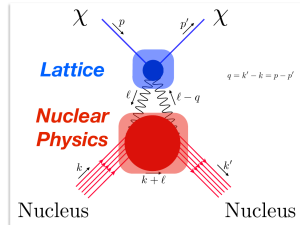
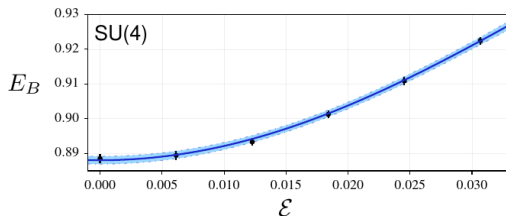
SU(4) gauge theory \rightarrow scalar dark baryon

No magnetic moment or charge radius,
Higgs exchange also suppressed



Polarizability places lower bound on direct-detection cross section

Compute on lattice as quadratic response to external field \mathcal{E}



Stealth dark matter direct detection

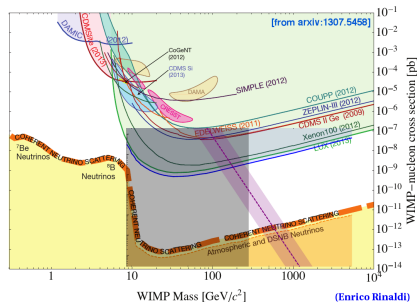
Polarizability places lower bound on direct-detection cross section

Not significantly constrained by existing underground experiments
and falls below coherent neutrino background for $M_{DM} \gtrsim 1$ TeV

Cross section specific to Xenon

Uncertainties dominated
by nuclear matrix element

(Similar matrix elements
arise in double-beta decay)



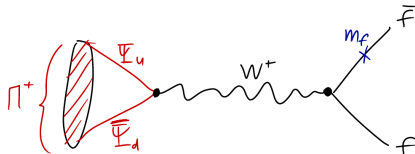
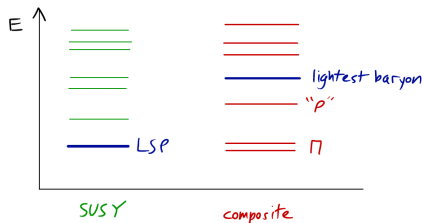
Shaded region is complementary constraint from collider searches

Stealth dark matter collider detection

Spectrum significantly different
from MSSM-inspired models

Very little missing E_T at colliders

Main constraints from
much lighter **charged** " Π " states



Rapid Π decays with $\Gamma \propto m_f^2$

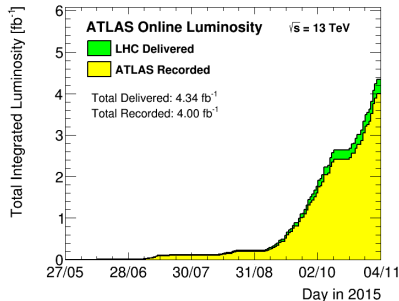
Best current constraints
recast stau searches at LEP

LHC can also search for $t\bar{b} + \bar{t}b$
from $\Pi^+\Pi^-$ Drell–Yan production

Application: Electroweak symmetry breaking

LHC experiments have collected
 $\sim 4 \text{ fb}^{-1}$ of data at $\sqrt{s} = 13 \text{ TeV}$

Soon we will see new constraints on
physics beyond the standard model
... and possibly new discoveries!



One compelling possibility is new strong dynamics
that produces a composite Higgs boson

Protects the electroweak scale from sensitivity to quantum effects
(solving the hierarchy / fine-tuning problem)

Lattice gauge theory has a crucial role to play
in exploring and understanding new strong dynamics

Composite Higgs vs. QCD

Electroweak symmetry breaking through new strong dynamics remains viable but must satisfy stringent experimental constraints

- The composite Higgs boson must have a mass of 125 GeV and standard-model-like properties
- Electroweak precision observables (e.g., S parameter) must be consistent with the standard model



If the new strong dynamics resembled QCD
these conditions would not be satisfied

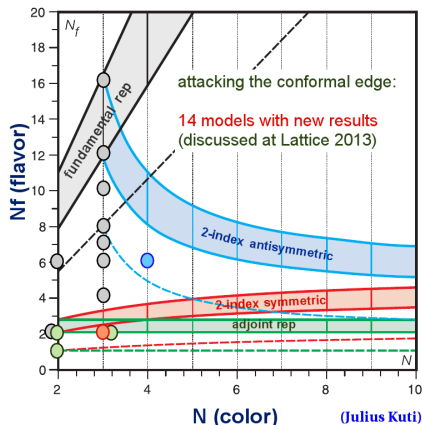
New strong dynamics different from QCD
can be studied non-perturbatively by lattice calculations

Strategy for lattice studies of new strong dynamics

Systematically depart from familiar ground of lattice QCD

($N = 3$ with $N_F = 2$ light flavors in fundamental rep)

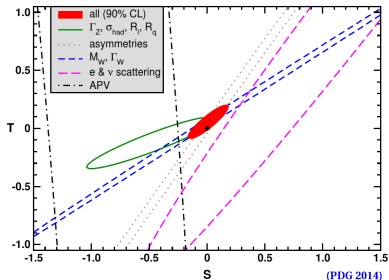
Explore the range of possible phenomena in strongly coupled theories



- Add more light flavors
→ $N_F = 8$ fundamental
- Enlarge fermion rep
→ $N_F = 2$ two-index symmetric
- Explore $N = 2$ and 4
→ (pseudo)real reps for cosets $SU(n)/Sp(n)$ and $SU(n)/SO(n)$

Electroweak precision observable — the S parameter

Corrections to vacuum polarization of neutral EW gauge bosons



S remains an important constraint
on new strong dynamics

Experiment: $S = 0.03 \pm 0.10$

Scaled-up QCD: $S \approx 0.43$

S is a low-energy constant (LEC) of electroweak chiral lagrangian \mathcal{L}_χ

Predicting LECs of low-energy effective theories

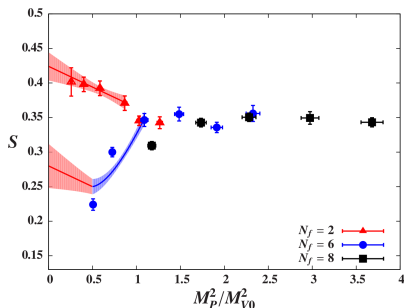
is a standard application of lattice gauge theory

The S parameter on the lattice

$$\mathcal{L}_\chi \supset \frac{\alpha_1}{2} g_1 g_2 B_{\mu\nu} \text{Tr} [U_{\tau_3} U^\dagger W^{\mu\nu}] \longrightarrow \gamma, Z \text{ } \text{new} \text{ } \gamma, Z$$

Lattice vacuum polarization calculation provides $S = -16\pi^2\alpha_1$

One subtlety is that nonzero masses needed
to keep correlation lengths insensitive to finite lattice volume

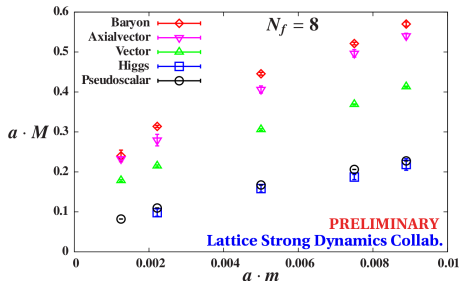


$S = 0.42(2)$ for $N_F = 2$
matches scaled-up QCD

Moving away from QCD with larger N_F
produces significant reductions

Extrapolation to correct zero-mass limit
becomes more challenging

The composite spectrum with eight flavors



Work in progress using
lattice volumes up to $64^3 \times 128$

Scale setting suggests
resonance masses $\sim 2\text{--}3$ TeV

Large separation between
Higgs and resonances

Higgs degenerate with pseudo-Goldstones in accessible regime

Dramatically different from QCD-like dynamics,

where $M_H \approx 2M_P$ in this regime (dominated by two-pion scattering)

Typical chiral extrapolation integrates out everything except pions,
can't reliably be applied to these data

Status of light composite Higgs from lattice

Without reliable chiral extrapolation we can only estimate

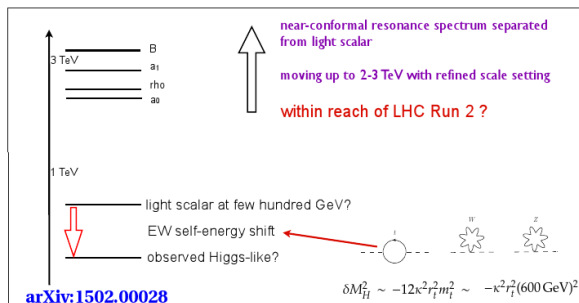
$$M_H \sim \text{few hundred GeV, with large error bars}$$

Much lighter than scaled-up QCD, still somewhat far from 125 GeV

Of course, we **shouldn't** get exactly 125 GeV

since we haven't yet incorporated electroweak & top corrections

These reduce M_H ,
but not yet consensus
on size of effect...



Application: Lattice supersymmetry

Supersymmetry is extremely interesting, especially non-perturbatively

- Widely studied potential roles in BSM physics
Central to ongoing LHC experimental program
→ current results greatly constrain simplest scenarios
- More generally, symmetries simplify systems
→ Insight into confinement, dynamical symmetry breaking, conformal field theories (exhibiting scale invariance), etc.
- Dualities: gauge–gauge (Seiberg) & gauge–gravity (AdS / CFT)
→ potential non-perturbative definition of string theory
- AdS / CFT-inspired modelling of quark–gluon plasma, finite-density phase diagram, condensed matter systems, etc.

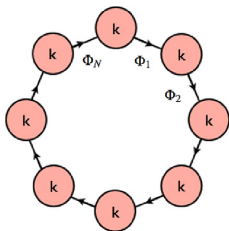
A brief history of lattice supersymmetry

Supersymmetry generalizes Poincaré symmetry, adding spinorial generators Q and \bar{Q} to translations, rotations, boosts

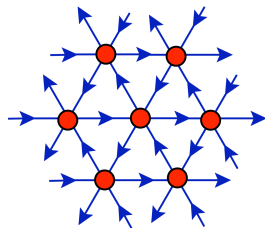
The algebra includes $Q\bar{Q} + \bar{Q}Q = 2\sigma^\mu P_\mu$,
broken in discrete space-time because no **infinitesimal translations**

Recent work overcomes this obstacle in certain contexts,
including maximally supersymmetric Yang–Mills (“ $\mathcal{N} = 4$ ” SYM)

Preserve supersymmetry **sub**algebra \implies recover rest in continuum



For details see
Catterall, Kaplan & Ünsal
[arXiv:0903.4881](https://arxiv.org/abs/0903.4881)



$\mathcal{N} = 4$ SYM — the fruit fly of QFT

Maximal supersymmetries make $\mathcal{N} = 4$ SYM
arguably the simplest non-trivial field theory in four dimensions

- $SU(N)$ gauge theory with four fermions Ψ^I and six scalars Φ^{IJ} ,
all massless and in adjoint rep.
- Action consists of kinetic, Yukawa and four-scalar terms
with coefficients related by symmetries
- Supersymmetric: 16 supercharges Q_α^I and $\bar{Q}_{\dot{\alpha}}^I$ with $I = 1, \dots, 4$
Fields and Q 's transform under global $SU(4) \simeq SO(6)$ symmetry
- Conformal: β function is zero for any 't Hooft coupling $\lambda = g^2 N$
(It is the conformal field theory of the first AdS / CFT duality)

$\mathcal{N} = 4$ SYM — the role of lattice gauge theory

$\mathcal{N} = 4$ SYM is widely studied by many different methods:

- Perturbation theory at weak coupling $\lambda \ll 1$,
related to strong coupling by S duality under $\frac{4\pi N}{\lambda} \longleftrightarrow \frac{\lambda}{4\pi N}$
- AdS / CFT holography for $N \rightarrow \infty$ and $\lambda \rightarrow \infty$ but $\lambda \ll N$
- Numerical optimization of conformal bootstrap relations

Only lattice gauge theory can access nonperturbative λ at moderate N

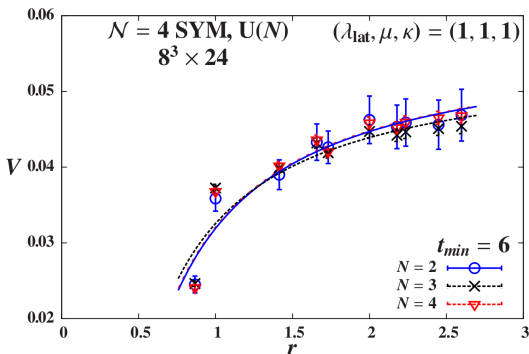
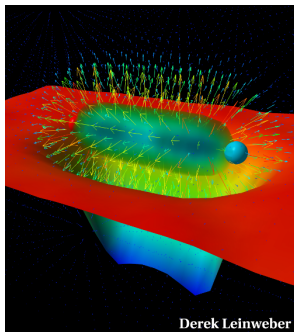
Let's see two results from ongoing lattice calculations

(work in progress with S. Catterall, P. Damgaard, J. Giedt & T. DeGrand)

- 1 Coupling dependence of the Coulomb potential
- 2 Scaling dimension of simplest conformal primary operator

Coulomb potential from lattice $\mathcal{N} = 4$ SYM

Lattice Wilson loops \rightarrow potential $V(r)$ between two static probes



Fits to confining $V(r) = A - C/r + \sigma r$

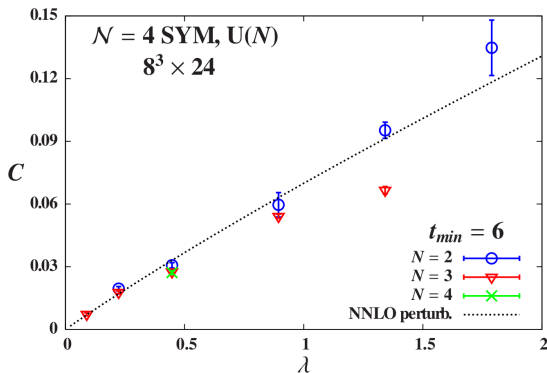
produce vanishing string tension $\sigma = 0$ for all couplings λ

Fits to Coulombic $V(r) = A - C/r$ predict Coulomb coefficient $C(\lambda)$

Coulomb coefficient from lattice $\mathcal{N} = 4$ SYM

Continuum perturbation theory predicts $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

AdS / CFT predicts $C(\lambda) \propto \sqrt{\lambda}$ for $N \rightarrow \infty$, $\lambda \rightarrow \infty$, $\lambda \ll N$



$N = 2$ results agree with perturbation theory for all $\lambda \lesssim N$

$N = 3$ results bend down for $\lambda \gtrsim 1$ — approaching AdS / CFT?

Konishi operator in lattice $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM is conformal for any coupling λ

→ power-law decay for all correlation functions $C(r) \propto r^{-2\Delta}$

The scaling dimension Δ_K of the simple Konishi operator
has attracted much recent attention

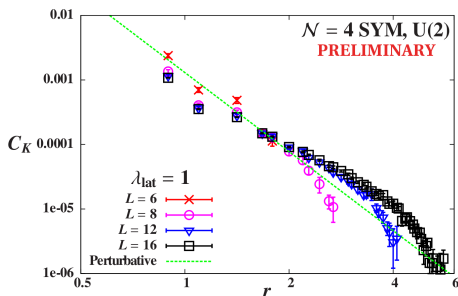
$$\mathcal{O}_K(x) = \sum_I \text{Tr} [\Phi^I(x) \Phi^I(x)] \quad (\text{symmetric sum over six scalars})$$

$$C_K(r) = \mathcal{O}_K(x+r) \mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

Sensitive to finite volume,
as desired for conformal system

Lattice tools to find Δ_K :

- Finite-size scaling
- Monte Carlo RG



Konishi scaling dimension from lattice $\mathcal{N} = 4$ SYM

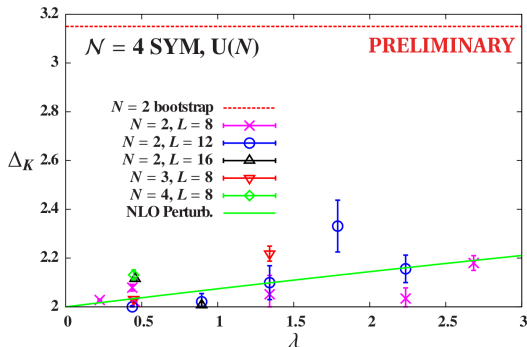
Recent studies of Konishi scaling dimension Δ_K

in perturbation theory + S duality, AdS / CFT holography, bootstrap

The first lattice calculation
of Δ_K is now underway

Plot shows initial results
from Monte Carlo RG
(only statistical errors)

Rough agreement
between $N = 2, 3, 4$



So far results follow perturbation theory, far from bootstrap bounds

Currently refining analyses, running larger volumes at stronger λ

Outlook: An exciting time for lattice gauge theory

- Continuing computational progress is enhancing applications across a broad range of physics
- Non-perturbative composite dark matter form factors place lower bound on direct-detection cross sections
- Lattice investigations of new strong dynamics find lighter Higgs and smaller S parameter than scaled-up QCD
- First large-scale lattice studies of $\mathcal{N} = 4$ SYM beginning to explore regimes inaccessible to other methods

Thank you!

Outlook: An exciting time for lattice gauge theory

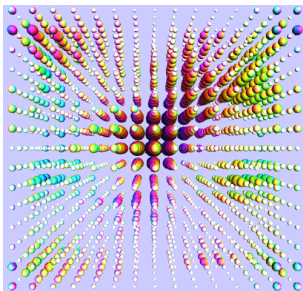
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Thank you!



Backup: Hybrid Monte Carlo (HMC) algorithm

Recall goal: Sample field configurations Φ_i with probability $\frac{1}{Z} e^{-S[\Phi_i]}$



HMC is a Markov process, based on
Metropolis–Rosenbluth–Teller (MRT)

Fermions \longrightarrow extensive action computation,
so best to update entire system at once

Use fictitious molecular dynamics evolution

- 1 Introduce a fictitious fifth dimension (“MD time” τ)
and stochastic canonical momenta for all field variables
- 2 Run inexact MD evolution along a trajectory in τ
to generate new four-dimensional field configuration
- 3 Apply MRT accept/reject test to MD discretization error

Backup:

Lattice Strong Dynamics Collaboration

Argonne Xiao-Yong Jin, James Osborn

Boston Rich Brower, Claudio Rebbi, Evan Weinberg

Brookhaven Meifeng Lin

Colorado Anna Hasenfratz, Ethan Neil

Edinburgh Oliver Witzel

Livermore Evan Berkowitz, Enrico Rinaldi, Pavlos Vranas

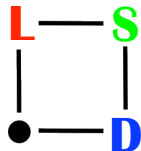
Oregon Graham Kribs

RBRC Ethan Neil, Sergey Syritsyn

Syracuse DS

UC Davis Joseph Kiskis

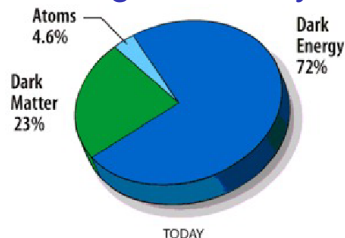
Yale Thomas Appelquist, George Fleming, Andy Gasbarro



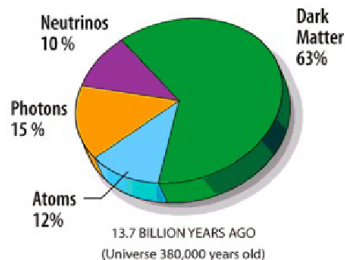
Exploring the range of possible phenomena
in strongly coupled gauge theories

Backup: Dark matter density in cosmological history

$$\frac{\Omega_{DM}}{\Omega_{baryons}} \approx \frac{\Omega_{DM}}{\Omega_{SM}} \approx 5 \text{ now}$$

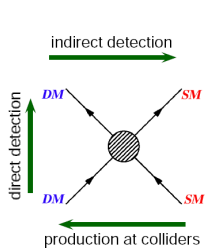


$$\frac{\Omega_{DM}}{\Omega_{baryons}} \approx 5 \text{ at recombination}$$



Simply because both are **matter** and evolve in the same way

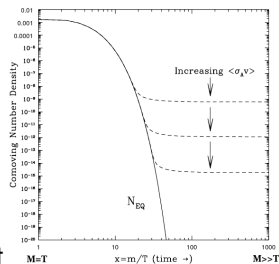
Backup: Thermal freeze-out for relic density



$T \gtrsim M_{DM}$: $DM \longleftrightarrow SM$
Thermal equilibrium

$T \lesssim M_{DM}$: $DM \rightarrow SM$
Rapid depletion of Ω_{DM}

Hubble expansion \rightarrow dilution
leads to freeze-out



Requires coupling between standard model and dark matter

Mass and coupling of **pure** thermal relic are related: $\frac{M_{DM}}{100 \text{ GeV}} \sim 200\alpha$

(The “WIMP miracle” is $\alpha \sim \alpha_{EW} \sim 0.01 \Rightarrow M_{DM} \sim 200 \text{ GeV} \sim v$)

Thermal relic suppressed by **strong** coupling, easy for composite DM

Backup: Two roads to natural asymmetric dark matter

Basic idea: Dark matter relic density related to baryon asymmetry

$$\Omega_D \approx 5\Omega_B$$
$$\implies M_D n_D \approx 5M_B n_B$$

- $n_D \sim n_B \implies M_D \sim 5M_B \approx 5 \text{ GeV}$

High-dimensional interactions relate baryon# and DM# violation

- $M_D \gg M_B \implies n_B \gg n_D \sim \exp[-M_D/T_s]$

Sphaleron transitions above $T_s \sim 200 \text{ GeV}$ distribute asymmetries

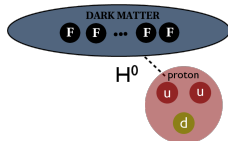
Both require coupling between standard model and dark matter

Backup: Effective Higgs interaction

Exchange of Higgs boson with $M_H = 125$ GeV

may dominate spin-independent direct detection cross section

$$\sigma_H^{(SI)} \propto \left| \frac{\mu_{B,N}}{M_H^2} y_\psi \langle B | \bar{\psi} \psi | B \rangle y_q \langle N | \bar{q} q | N \rangle \right|^2$$



For **quarks** $y_q = \frac{m_q}{v} \Rightarrow y_q \langle N | \bar{q} q | N \rangle \propto \frac{M_N}{v} \frac{\langle N | m_q \bar{q} q | N \rangle}{M_N}$

For **dark constituent fermions** ψ

there is an additional model parameter, $y_q = \alpha \frac{m_\psi}{v}$

In both cases the scalar form factor is most easily determined

using the Feynman–Hellmann theorem

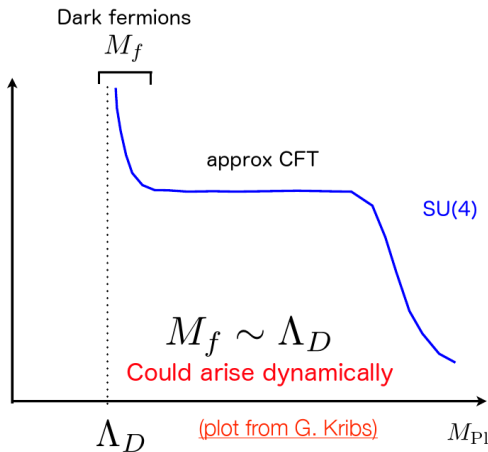
$$\frac{\langle B | m_\psi \bar{\psi} \psi | B \rangle}{M_B} = \frac{m_\psi}{M_B} \frac{\partial M_B}{\partial m_\psi}$$

Backup: Stealth dark matter mass scales

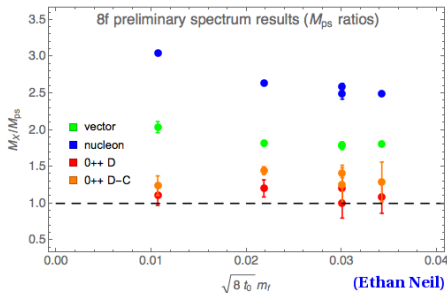
Lattice calculations have focused on $m_\psi \simeq \Lambda_D$,
the regime where analytic estimates are least reliable

This mass scale has
some theoretical motivation

In addition,
collider constraints tighten
as mass decreases



Backup: Eight-flavor spectrum in dimensionless ratios



Work in progress using
lattice volumes up to $64^3 \times 128$

Scale setting suggests
resonance masses $\sim 2-3$ TeV

Large separation between
Higgs and resonances

Higgs degenerate with pseudo-Goldstones in accessible regime

Dramatically different from QCD-like dynamics,

where $M_H \approx 2M_P$ in this regime (dominated by two-pion scattering)

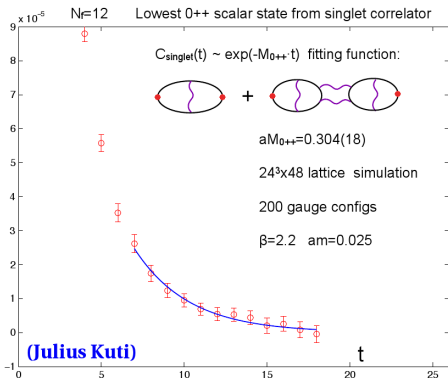
Typical chiral extrapolation integrates out everything except pions,
can't reliably be applied to these data

Backup: Technical lattice challenge for Higgs state

Only the new strong sector is included in the lattice calculation

⇒ The Higgs is a singlet that mixes with the vacuum

Leads to noisy data and relatively large uncertainties in Higgs mass



Fermion propagator computation
is relatively expensive

“Disconnected diagrams” formally
need propagators at all L^4 sites

In practice compute stochastically
to control computational costs

Backup: Vacuum polarization is just current correlator

$$S = 4\pi N_D \lim_{Q^2 \rightarrow 0} \frac{d}{dQ^2} \Pi_{V-A}(Q^2) - \Delta S_{SM}(M_H)$$



$$\Pi_{V-A}^{\mu\nu}(Q) = Z \sum_x e^{iQ \cdot (x + \hat{\mu}/2)} \text{Tr} \left[\langle \mathcal{V}^{\mu a}(x) V^{\nu b}(0) \rangle - \langle \mathcal{A}^{\mu a}(x) A^{\nu b}(0) \rangle \right]$$

$$\Pi^{\mu\nu}(Q) = \left(\delta^{\mu\nu} - \frac{\hat{Q}^\mu \hat{Q}^\nu}{\hat{Q}^2} \right) \Pi(Q^2) - \frac{\hat{Q}^\mu \hat{Q}^\nu}{\hat{Q}^2} \Pi^L(Q^2) \quad \hat{Q} = 2 \sin(Q/2)$$

- Renormalization constant Z evaluated non-perturbatively
Chiral symmetry of domain wall fermions $\implies Z = Z_A = Z_V$
 $Z = 0.85$ [2f]; 0.73 [6f]; 0.70 [8f]
- Conserved currents \mathcal{V} and \mathcal{A} ensure that lattice artifacts cancel

Backup: Failure of Leibnitz rule in discrete space-time

Given that $\left\{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \right\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu = 2i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu$ is problematic,

why not try $\left\{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \right\} = 2i\sigma_{\alpha\dot{\alpha}}^\mu \nabla_\mu$ for a discrete translation?

Here $\nabla_\mu \phi(x) = \frac{1}{a} [\phi(x + a\hat{\mu}) - \phi(x)] = \partial_\mu \phi(x) + \frac{a}{2} \partial_\mu^2 \phi(x) + \mathcal{O}(a^2)$

Essential difference between ∂_μ and ∇_μ on the lattice, $a > 0$

$$\begin{aligned}\nabla_\mu [\phi(x)\chi(x)] &= a^{-1} [\phi(x + a\hat{\mu})\chi(x + a\hat{\mu}) - \phi(x)\chi(x)] \\ &= [\nabla_\mu \phi(x)] \chi(x) + \phi(x) \nabla_\mu \chi(x) + a [\nabla_\mu \phi(x)] \nabla_\mu \chi(x)\end{aligned}$$

We only recover the Leibnitz rule $\partial_\mu(fg) = (\partial_\mu f)g + f\partial_\mu g$ when $a \rightarrow 0$
 \implies “Discrete supersymmetry” breaks down on the lattice

(Dondi & Nicolai, “Lattice Supersymmetry”, 1977)

Backup: Potential sign problem of $\mathcal{N} = 4$ SYM

Integrating over a single Kähler–Dirac fermion Ψ in adjoint rep.,

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [dU][d\bar{U}] \mathcal{O} e^{-S_B[U, \bar{U}]} \text{pf } \mathcal{D}[U, \bar{U}]$$

Pfaffian can be complex for lattice $\mathcal{N} = 4$ SYM, $\text{pf } \mathcal{D} = |\text{pf } \mathcal{D}| e^{i\alpha}$

Complicates interpretation of $\{e^{-S_B} \text{pf } \mathcal{D}\}$ as Boltzmann weight

We carry out phase-quenched RHMC, $\text{pf } \mathcal{D} \rightarrow |\text{pf } \mathcal{D}|$

In principle need to reweight phase-quenched (pq) observables:

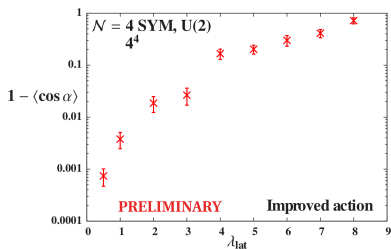
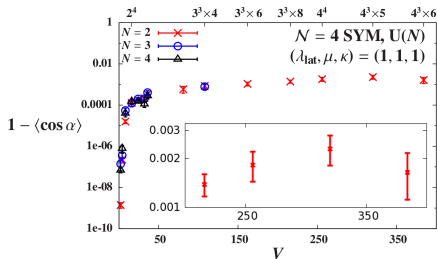
$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}} \quad \text{with} \quad \langle \mathcal{O} e^{i\alpha} \rangle_{pq} = \frac{1}{\mathcal{Z}_{pq}} \int [dU][d\bar{U}] \mathcal{O} e^{i\alpha} e^{-S_B} |\text{pf } \mathcal{D}|$$

\Rightarrow Monitor $\langle e^{i\alpha} \rangle_{pq}$ as function of volume, coupling, N

Backup: Pfaffian dependence on volume and coupling

Left: $1 - \langle \cos(\alpha) \rangle_{pq} \ll 1$ independent of volume and N at $\lambda_{\text{lat}} = 1$

Right: New 4^4 results at $4 \leq \lambda_{\text{lat}} \leq 8$ show much larger fluctuations



Currently filling in more volumes and N for improved action

Extremely expensive analysis despite new parallel algorithm:

$\mathcal{O}(n^3)$ scaling $\rightarrow \sim 50$ hours for single 4^4 measurement

Backup: Two puzzles posed by the sign problem

- With **periodic temporal boundary conditions** for the fermions we have an obvious sign problem, $\langle e^{i\alpha} \rangle_{pq}$ consistent with zero
- With **anti-periodic BCs** and all else the same $e^{i\alpha} \approx 1$, phase reweighting has negligible effect

Why such sensitivity to the BCs?

Also, other observables
are nearly identical
for these two ensembles

Why doesn't the sign problem
affect other observables?

