$\mathcal{N} = 4$ supersymmetric Yang–Mills on a space-time lattice

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arXiv:1411.0166, arXiv:1505.03135, arXiv:1508.00884 & more to come
with Simon Catterall, Poul Damgaard, Tom DeGrand and Joel Giedt

Plan

- Motivations for lattice supersymmetry in general
- Lattice formulation of $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM)
  [new improvement procedure & public code]
- Latest results for static potential and Konishi anomalous dim.
  [confront with perturbation theory, AdS/CFT, bootstrap]
- Prospects and future directions
  [sign problem; lattice superQCD in two & three dimensions]

Motivation: Why lattice supersymmetry

Lattice discretization provides non-perturbative,
gauge-invariant regularization of vectorlike gauge theories

Amenable to first-principles numerical analysis
$\longrightarrow$ complementary approach to study strongly coupled field theories

Proven success for QCD; many potential susy applications:

- Compute Wilson loops, spectrum, scaling dimensions, etc., going beyond perturbation theory, holography, bootstrap, ...
- New non-perturbative tests of conjectured dualities
- Predict low-energy constants from dynamical susy breaking
- Validate or refine AdS/CFT-based models for QCD phase diagram, condensed matter systems, ...

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Many ideas probably infeasible; relatively few have been explored

Obstruction: Why not lattice supersymmetry

The super-Poincaré algebra includes $\left\{Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{J}}_{\dot{\alpha}}\right\} = 2\delta^{\mathrm{IJ}}\sigma^{\mu}_{\alpha\dot{\alpha}}P_{\mu}$

but infinitesimal translations don't exist in discrete space-time

Consequences for lattice calculations

Explicitly broken supersymmetry $\Longrightarrow$ relevant susy-violating operators

Typically many such operators,
especially with scalar fields from matter multiplets or from $\mathcal{N} > 1$

Fine-tuning couplings / counterterms to restore supersymmetry
is generally not practical in numerical lattice calculations

Solution: Exact susy on the lattice

Rapid progress in recent years

In certain systems some subset of the susy algebra
can be exactly preserved at non-zero lattice spacing

Equivalent constructions obtained from orbifolding / deconstruction
and from "topological" twisting — cf. arXiv:0903.4881 for review

In 4D these constructions pick out a unique system: $\mathcal{N} = 4$ SYM

- SU(N) gauge theory with four fermions Ψ^{I} and six scalars Φ^{IJ}, all massless and in adjoint rep.
- Global SU(4) $\simeq$ SO(6) R symmetry
- 16 supercharges Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$
- Conformal: β function is zero for any 't Hooft coupling $\lambda = g^2 N$

What is special about $\mathcal{N} = 4$ SYM

Intuitive picture of Geometric-Langlands twist for $\mathcal{N} = 4$ SYM

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu\gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu\gamma_\mu\gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a\gamma_a + \mathcal{Q}_{ab}\gamma_a\gamma_b$$

with $a, b = 1, \cdots, 5$

$\mathcal{Q}$'s transform with **integer spin** under "twisted rotation group"

$$\mathrm{SO}(4)_{tw} \equiv \mathrm{diag}\left[\mathrm{SO}(4)_{\mathrm{euc}} \otimes \mathrm{SO}(4)_R\right] \qquad \mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$$

This change of variables gives a susy subalgebra $\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$

This subalgebra can be exactly preserved on the lattice

Twisted $\mathcal{N} = 4$ SYM fields and $\mathcal{Q}$

Everything transforms with **integer spin** under $\mathrm{SO}(4)_{tw}$ — **no spinors**

$$Q^{\mathrm{I}}_{\alpha} \text{ and } \overline{Q}^{\mathrm{I}}_{\dot{\alpha}} \longrightarrow \mathcal{Q},\ \mathcal{Q}_a \text{ and } \mathcal{Q}_{ab}$$

$$\Psi^{\mathrm{I}} \text{ and } \overline{\Psi}^{\mathrm{I}} \longrightarrow \eta,\ \psi_a \text{ and } \chi_{ab}$$

$$A_{\mu} \text{ and } \Phi^{\mathrm{IJ}} \longrightarrow \mathcal{A}_a = (A_{\mu}, \phi) + i(B_{\mu}, \overline{\phi}) \text{ and } \overline{\mathcal{A}}_a$$

Complexify gauge fields since scalars $\longrightarrow$ vectors under twisting
(complexification $\Longrightarrow$ $\mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1)$ gauge invariance)

The motivation is most obvious in five dimensions where

$$\mathrm{SO}(5)_{tw} \equiv \mathrm{diag}\left[\mathrm{SO}(5)_{\mathrm{euc}} \otimes \mathrm{SO}(5)_R\right]$$

Then dimensional reduction takes gauge fields $A_a \to (A_{\mu}, \phi)$
and scalar fields $B_a \to (B_{\mu}, \overline{\phi})$

Twisted $\mathcal{N} = 4$ SYM fields and $\mathcal{Q}$

Everything transforms with **integer spin** under $\text{SO}(4)_{tw}$ **— no spinors**

$$Q^{\text{I}}_{\alpha} \;\text{ and }\; \overline{Q}^{\text{I}}_{\dot\alpha} \longrightarrow \mathcal{Q},\; \mathcal{Q}_a \;\text{ and }\; \mathcal{Q}_{ab}$$

$$\Psi^{\text{I}} \;\text{ and }\; \overline{\Psi}^{\text{I}} \longrightarrow \eta,\; \psi_a \;\text{ and }\; \chi_{ab}$$

$$A_\mu \;\text{ and }\; \Phi^{\text{IJ}} \longrightarrow \mathcal{A}_a = (A_\mu, \phi) + i(B_\mu, \overline{\phi}) \;\text{ and }\; \overline{\mathcal{A}}_a$$

The twisted-scalar supersymmetry $\mathcal{Q}$

correctly interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\begin{aligned} &\mathcal{Q}\,\mathcal{A}_a = \psi_a & &\mathcal{Q}\,\psi_a = 0 \\ &\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} & &\mathcal{Q}\,\overline{\mathcal{A}}_a = 0 \\ &\mathcal{Q}\,\eta = d & &\mathcal{Q}\,d = 0 \end{aligned}$$

$\nwarrow$ bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

Lattice $\mathcal{N} = 4$ SYM

The lattice theory is nearly a direct transcription,

despite breaking the 15 $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

- Covariant derivatives $\longrightarrow$ finite difference operators
- Complexified gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $\mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C})$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$
$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$
$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

Geometry manifest: η and d on sites, $\mathcal{U}_a$ and ψ_a on links, etc.

- Supersymmetric lattice action ($\mathcal{Q}S = 0$)

follows from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity

$$S = \frac{N}{2\lambda_{\text{lat}}} \mathcal{Q} \left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde}\, \chi_{ab}\overline{\mathcal{D}}_c\, \chi_{de}$$

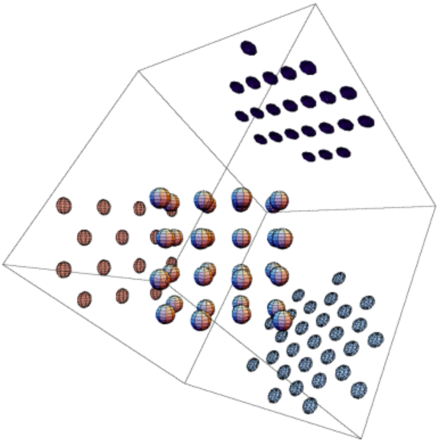
Five links in four dimensions $\longrightarrow A_4^*$ lattice

Revisit dimensional reduction in discrete spacetime,
treating all five $\mathcal{U}_a$ symmetrically

—Start with hypercubic lattice
in 5D momentum space

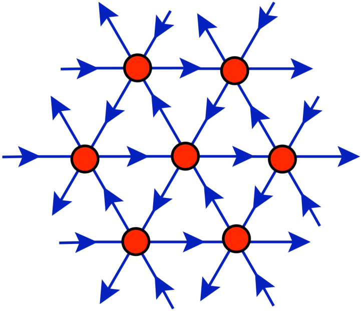
—**Symmetric** constraint $\sum_a \partial_a = 0$
projects to 4D momentum space

—Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in real space



Twisted SO(4) symmetry on the A_4^* lattice

—Can picture A_4^* lattice
as 4D analog of 2D triangular lattice



—Basis vectors are linearly dependent
and non-orthogonal $\longrightarrow \lambda = \lambda_{\text{lat}}/\sqrt{5}$

—Preserves S_5 point group symmetry

S_5 irreps precisely match onto irreps of twisted $\text{SO}(4)_{tw}$

$$\mathbf{5} = \mathbf{4} \oplus \mathbf{1} : \quad \psi_a \longrightarrow \psi_\mu, \ \ \overline{\eta}$$

$$\mathbf{10} = \mathbf{6} \oplus \mathbf{4} : \quad \chi_{ab} \longrightarrow \chi_{\mu\nu}, \ \ \overline{\psi}_\mu$$

$S_5 \longrightarrow \text{SO}(4)_{tw}$ in continuum limit restores the rest of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Twisted $\mathcal{N} = 4$ SYM on the A_4^* lattice

High degree of exact lattice symmetry: gauge invariance + $\mathcal{Q}$ + S_5

Several important analytic consequences:

- Moduli space preserved to all orders of lattice perturbation theory $\longrightarrow$ no scalar potential induced by radiative corrections
- β function vanishes at one loop in lattice perturbation theory
- Real-space RG blocking transformations preserve $\mathcal{Q}$ and S_5
- Only one log. tuning to recover $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ in the continuum

Not quite suitable for numerical calculations

Exact zero modes and flat directions must be regulated, especially important in U(1) sector

$\mathcal{N} = 4$ SYM lattice action (I)

$$S = \frac{N}{2\lambda_{\text{lat}}} \mathcal{Q} \left(\chi_{ab}\mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a \mathcal{U}_a - \frac{1}{2}\eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde}\ \chi_{ab} \overline{\mathcal{D}}_c\ \chi_{de} + \mu^2 V$$

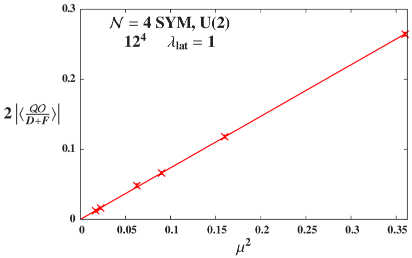
Scalar potential $V = \frac{1}{2N\lambda_{\text{lat}}} \left(\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - N\right)^2$ lifts SU(N) flat directions and ensures $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

Breaks $\mathcal{Q}$ **softly** — susy breaking automatically vanishes as $\mu^2 \to 0$

Violations of Ward identities $\langle \mathcal{Q}\mathcal{O} \rangle = 0$ show $\mathcal{Q}$ breaking and restoration

Here considering

$$\mathcal{Q}\left[\eta \mathcal{U}_a \overline{\mathcal{U}}_a\right] = d\mathcal{U}_a\overline{\mathcal{U}}_a - \eta\psi_a\overline{\mathcal{U}}_a$$



Full $\mathcal{N} = 4$ SYM lattice action [arXiv:1505.03135]

$$S = \frac{N}{2\lambda_{\text{lat}}} \mathcal{Q} \left(\chi_{ab}\mathcal{F}_{ab} + \quad \downarrow \quad - \frac{1}{2}\eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde}\ \chi_{ab}\overline{\mathcal{D}}_c\ \chi_{de} + \mu^2 V$$

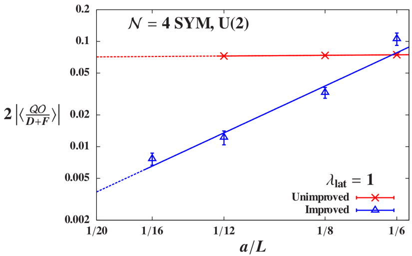
$$\eta \left(\overline{\mathcal{D}}_a \mathcal{U}_a + G \sum_{a<b} \left[\det \mathcal{P}_{ab} - 1\right] \mathbb{I}_N \right)$$

Constraint on plaquette det. lifts U(1) zero mode & flat directions

$\mathcal{Q}$-exact implementation as new moduli space condition

Leads to $\langle \mathcal{Q}\mathcal{O} \rangle \propto (a/L)^2$,
much better than naive constraint

Effective "$\mathcal{O}(a)$ improvement"
since $\mathcal{Q}$ forbids all dim-5 operators



Advertisement: Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$
\begin{aligned}
S_{\text{imp}} &= S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \qquad (3.10)\\
S'_{\text{exact}} &= \frac{N}{2\lambda_{\text{lat}}}\sum_n \text{Tr}\left[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_a\psi_a(n)\right.\\
&\qquad\left. + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a(n) + G\sum_{a\neq b}\left(\det\mathcal{P}_{ab}(n) - 1\right)\mathbb{I}_N\right)^2\right] - S_{\text{det}}\\
S_{\text{det}} &= \frac{N}{2\lambda_{\text{lat}}}G\sum_n \text{Tr}\left[\eta(n)\right]\sum_{a\neq b}\left[\det\mathcal{P}_{ab}(n)\right]\text{Tr}\left[\mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n+\widehat{\mu}_b)\psi_a(n+\widehat{\mu}_b)\right]\\
S_{\text{closed}} &= -\frac{N}{8\lambda_{\text{lat}}}\sum_n \text{Tr}\left[\epsilon_{abcde}\ \chi_{de}(n+\widehat{\mu}_a+\widehat{\mu}_b+\widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_c\chi_{ab}(n)\right],\\
S'_{\text{soft}} &= \frac{N}{2\lambda_{\text{lat}}}\mu^2\sum_n\sum_a\left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right)^2
\end{aligned}
$$

The full $\mathcal{N} = 4$ SYM lattice action is somewhat complicated

(For experts: $\gtrsim$100 inter-node data transfers in the fermion operator)

To reduce barriers to entry our parallel code is publicly developed at

github.com/daschaich/susy

Evolved from MILC lattice QCD code, presented in arXiv:1410.6971

Application: Static potential (Coulombic at all λ)

Static potential $V(r)$ from $r \times T$ Wilson loops: $W(r, T) \propto e^{-V(r)\,T}$

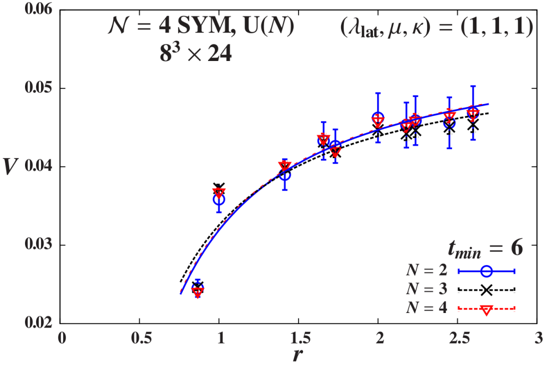
Fit $V(r)$ to Coulombic or confining form

$$V(r) = A - C/r$$

$$V(r) = A - C/r + \sigma r$$

C is Coulomb coefficient

σ is string tension



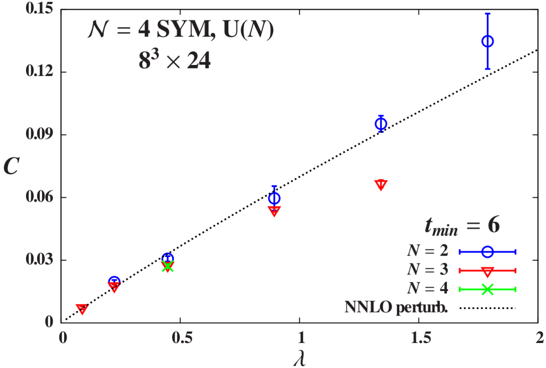
Fits to confining form always produce vanishing string tension $\sigma = 0$

More sophisticated analyses in development using improved action

Coupling dependence of Coulomb coefficient

Continuum perturbation theory predicts $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

AdS/CFT predicts $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$, $\lambda \to \infty$, $\lambda \ll N$



$N = 2$ results agree with perturbation theory for all $\lambda \lesssim N$

$N = 3$ results bend down for $\lambda \gtrsim 1$ — approaching AdS/CFT?

Application: Konishi operator scaling dimension

$\mathcal{N} = 4$ SYM is conformal at all λ
$\longrightarrow$ power-law decay for all correlation functions

The Konishi operator is the simplest conformal primary operator

$$\mathcal{O}_K(x) = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}(x)\Phi^{\mathrm{I}}(x)\right] \qquad C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

There are many predictions for its scaling dim. $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$

- From weak-coupling perturbation theory,
  related to strong coupling by $\frac{4\pi N}{\lambda} \longleftrightarrow \frac{\lambda}{4\pi N}$ S duality
- From holography for $N \to \infty$ and $\lambda \to \infty$ but $\lambda \ll N$
- Upper bounds from the conformal bootstrap program

Only lattice gauge theory can access nonperturbative λ at moderate N

Konishi operator on the lattice

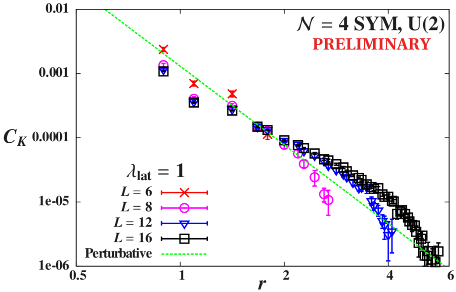
Extract scalar fields from polar decomposition of complexified links

$$\mathcal{U}_a \simeq (\mathbb{I}_N + \varphi_a)\, U_a \qquad \widehat{\mathcal{O}}_K = \sum_a \mathrm{Tr}\left[\varphi_a \varphi_a\right] \qquad \overline{\mathcal{O}}_K = \widehat{\mathcal{O}}_K - \left\langle \widehat{\mathcal{O}}_K \right\rangle$$

$$\overline{C}_K(r) = \overline{\mathcal{O}}_K(x+r)\overline{\mathcal{O}}_K(x) \propto r^{-2\Delta_K}$$

Sensitive to finite volume,
 as desired for conformal system

Good lattice tools to find Δ_K:
 —Finite-size scaling
 —Monte Carlo RG



Need lattice RG blocking scheme to carry out MCRG...

Real-space RG for lattice $\mathcal{N} = 4$ SYM

Lattice RG blocking transformation must preserve symmetries
$\mathcal{Q}$ and $S_5 \longleftrightarrow$ geometric structure of the system

Simple scheme constructed in arXiv:1408.7067:

$$\mathcal{U}'_a(x') = \xi\, \mathcal{U}_a(x)\mathcal{U}_a(x + \widehat{\mu}_a) \qquad \eta'(x') = \eta(x)$$
$$\psi'_a(x') = \xi \left[\psi_a(x)\mathcal{U}_a(x + \widehat{\mu}_a) + \mathcal{U}_a(x)\psi_a(x + \widehat{\mu}_a)\right] \qquad \text{etc.}$$

Doubles lattice spacing $a \longrightarrow a' = 2a$, with ξ a tunable rescaling factor

Set ξ by equating plaquette on n-times-blocked L^4 ensemble
with that on independent $(n-1)$-times-blocked $(L/2)^4$ ensemble

$\mathcal{Q}$-preserving RG blocking is necessary ingredient in derivation that
only one log. tuning needed to recover $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ in the continuum

Scaling dimensions from MCRG stability matrix

Write system as (infinite) sum of operators, $H = \sum_i c_i \, \mathcal{O}_i$

with couplings c_i that flow under RG blocking transformation R_b

n-times-blocked system is $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Fixed point defined by $H^\star = R_b H^\star$ with couplings $c_i^\star$

Linear expansion around fixed point defines stability matrix $T_{ij}^\star$

$$c_i^{(n)} - c_i^\star = \sum_j \left. \frac{\partial c_i^{(n)}}{\partial c_j^{(n-1)}} \right|_{H^\star} \left(c_j^{(n-1)} - c_j^\star \right) \equiv \sum_j T_{ij}^\star \left(c_j^{(n-1)} - c_j^\star \right)$$

Correlators of $\mathcal{O}_i$, $\mathcal{O}_j \longrightarrow$ elements of stability matrix (Swendsen, 1979)

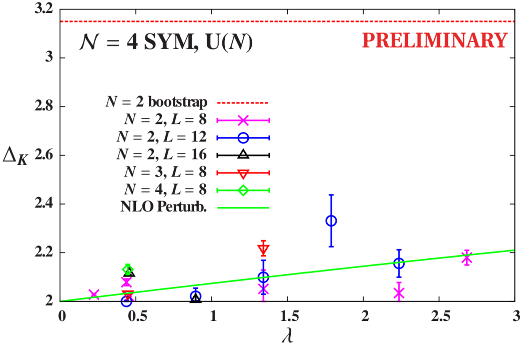
Eigenvalues of $T_{ij}^\star \longrightarrow$ scaling dimensions of corresponding operators

Preliminary Δ_K results from Monte Carlo RG

Far from bootstrap bounds

Rough agreement
between $N = 2, 3, 4$

Aim to distinguish
perturbative vs. free Δ_K



Only statistical uncertainties so far, averaged over

- ⋆ 1 & 2 RG blocking steps
- ⋆ Blocked volumes 3^4 through 8^4
- ⋆ 1–5 operators in stability matrix

More sophisticated analyses in development,
while running larger volumes at stronger couplings

Practical question: Potential sign problem

In lattice gauge theory we compute operator expectation values

$$\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}}\int[d\mathcal{U}][d\overline{\mathcal{U}}]\ \mathcal{O}\ \ e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]}\ \ \text{pf}\,\mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

Pfaffian can be complex for lattice $\mathcal{N}=4$ SYM, $\text{pf}\,\mathcal{D} = |\text{pf}\,\mathcal{D}|e^{i\alpha}$

Complicates interpretation of $\left\{e^{-S_B}\,\text{pf}\,\mathcal{D}\right\}$ as Boltzmann weight

We carry out phase-quenched RHMC, $\text{pf}\,\mathcal{D} \longrightarrow |\text{pf}\,\mathcal{D}|$

In principle need to reweight phase-quenched (pq) observables:

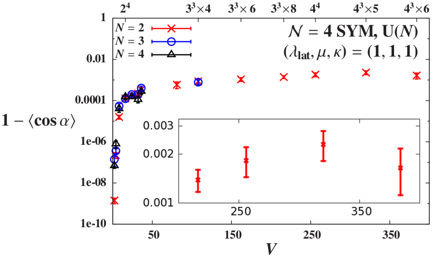
$$\langle\mathcal{O}\rangle = \frac{\left\langle\mathcal{O}e^{i\alpha}\right\rangle_{pq}}{\left\langle e^{i\alpha}\right\rangle_{pq}} \qquad \text{with}\ \left\langle\mathcal{O}e^{i\alpha}\right\rangle_{pq} = \frac{1}{\mathcal{Z}_{pq}}\int[d\mathcal{U}][d\overline{\mathcal{U}}]\,\mathcal{O}e^{i\alpha}\,e^{-S_B}\,|\text{pf}\,\mathcal{D}|$$

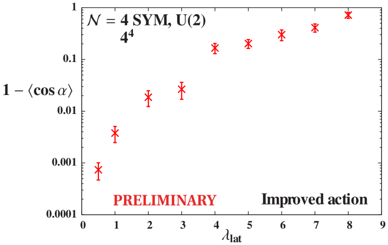
$\Longrightarrow$ Monitor $\left\langle e^{i\alpha}\right\rangle_{pq}$ as function of volume, coupling, N

Pfaffian phase dependence on volume and coupling

Left: $1 - \langle\cos(\alpha)\rangle_{pq} \ll 1$ independent of volume and N at $\lambda_{\text{lat}} = 1$

Right: New 4^4 results at $4 \leq \lambda_{\text{lat}} \leq 8$ show much larger fluctuations





Currently filling in more volumes and N for improved action

Extremely expensive analysis despite new parallel algorithm:

$\mathcal{O}(n^3)$ scaling $\longrightarrow$ ~50 hours for single 4^4 measurement

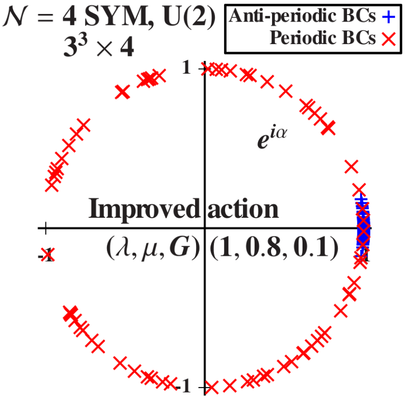
Two puzzles posed by the sign problem

- With periodic temporal boundary conditions for the fermions
  we have an obvious sign problem, $\left\langle e^{i\alpha} \right\rangle_{pq}$ consistent with zero
- With anti-periodic BCs and all else the same $e^{i\alpha} \approx 1$,
  phase reweighting has negligible effect

Why such sensitivity to the BCs?

Also, other pq observables
are nearly identical
for these two ensembles

Why doesn't the sign problem
affect other observables?



Preview: $(d-1)$-dimensional lattice superQCD

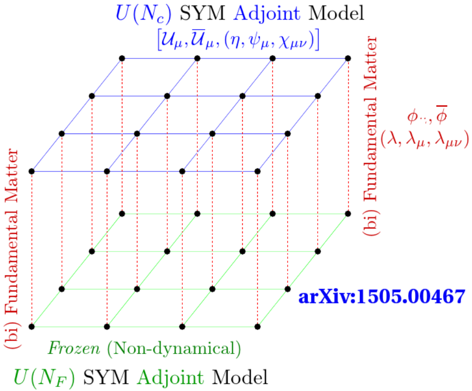
Method to add fundamental matter multiplets without breaking $\mathcal{Q}^2 = 0$

—Proposed by Matsuura (arXiv:0805.4491), Sugino (arXiv:0807.2683)
—First numerical study by Catterall & Veernala, arXiv:1505.00467

Consider 2-slice lattice
with U(N)$\times$U(F) gauge group:
—(Adj, 1) fields on one slice
—(1, Adj) fields on the other
—Bi-fund. ($\square$, $\overline{\square}$) in between

Set U(F) gauge coupling to zero
$\longrightarrow$ U(N) in $d-1$ dims.
with F fund. hypermultiplets

(Periodic BC $\longrightarrow$ anti-fund.)



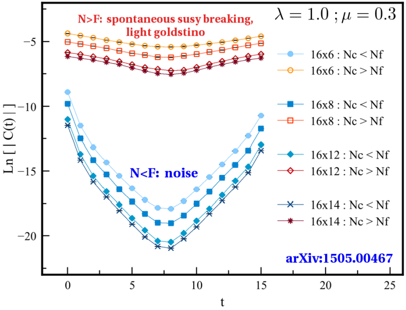
Spontaneous supersymmetry breaking

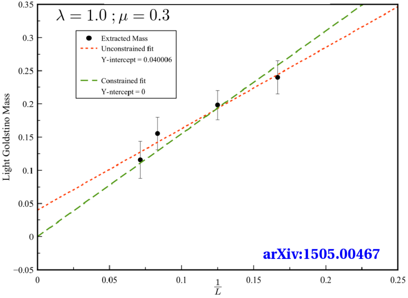
Can add $\mathcal{Q}$-exact moduli space condition (Fayet–Iliopoulos D term),

$$\eta\left(\overline{\mathcal{D}}_\mu \mathcal{U}_\mu + \sum_{i=1}^{F} \phi_i \overline{\phi}_i\right) \quad \longrightarrow \quad \eta\left(\overline{\mathcal{D}}_\mu \mathcal{U}_\mu + \sum_{i=1}^{F} \phi_i \overline{\phi}_i + r\mathbb{I}_N\right)$$

$\langle d \rangle = \left\langle \sum_{i=1}^{F} \phi_i \overline{\phi}_i + r\mathbb{I}_N \right\rangle$ and $\langle d \rangle \neq 0 \Longrightarrow$ spontaneous susy breaking

Effectively $N \times N$ conditions imposed on $N \times F$ degrees of freedom...





Recapitulation and outlook

Rapid recent progress in lattice supersymmetry

- Lattice promises non-perturbative insights from first principles
- Lattice $\mathcal{N} = 4$ SYM is practical thanks to exact $\mathcal{Q}$ susy
- Public code to reduce barriers to entry

Latest results from ongoing calculations

- Static potential is Coulombic at all couplings,
  $C(\lambda)$ confronted with perturbation theory and AdS/CFT
- Promising initial Konishi anomalous dimension at weak coupling

Many more directions are being — or can be — pursued

- Understanding the (absence of a) sign problem
- Systems with less supersymmetry, in lower dimensions,
  including matter fields, exhibiting spontaneous susy breaking, ...

Thank you!

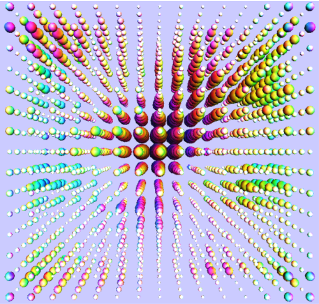
Thank you!

Collaborators

Simon Catterall, Poul Damgaard, Tom DeGrand and Joel Giedt

Funding and computing resources

Backup: Essence of numerical lattice calculations



Evaluate observables from functional integral via importance sampling Monte Carlo

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U \;\; \mathcal{O}(U) \;\; e^{-S[U]}$$

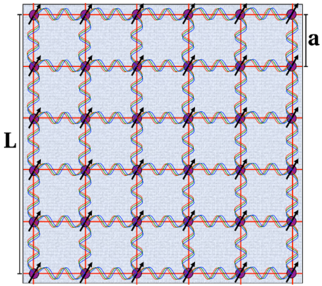
$$\longrightarrow \frac{1}{N} \sum_{n=1}^{N} \mathcal{O}(U_n) \pm \text{err} \left(\sqrt{\frac{1}{N}} \right)$$

U are field configurations in discretized euclidean spacetime

$S[U]$ is the lattice action, which should be real and positive so that $\frac{1}{\mathcal{Z}} e^{-S}$ can be treated as a probability distribution

The hybrid Monte Carlo algorithm samples U with probability $\propto e^{-S}$

Backup: More features of lattice calculations



Spacing between lattice sites (“a”)
introduces UV cutoff scale $1/a$

Lattice cutoff preserves hypercubic subgroup
of full Lorentz symmetry

Remove cutoff by taking
continuum limit $a \to 0$ (with $L/a \to \infty$)

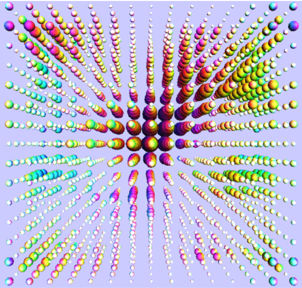
The lattice action S is defined by the bare lagrangian
at the UV cutoff set by the lattice spacing

After generating and saving an ensemble $\{U_n\}$ distributed $\propto e^{-S}$
it is usually quick and easy to measure many observables $\langle \mathcal{O} \rangle$

Changing the action (generally) requires generating a new ensemble

Backup: Hybrid Monte Carlo (HMC) algorithm

Goal: Sample field configurations U_i with probability $\frac{1}{\mathcal{Z}} e^{-S[U_i]}$



HMC is a Markov process, based on Metropolis–Rosenbluth–Teller (MRT)

Fermions $\longrightarrow$ extensive action computation, so best to update entire system at once

Use fictitious molecular dynamics evolution

1. Introduce a fictitious fifth dimension ("MD time" τ) and stochastic canonical momenta for all field variables
2. Run inexact MD evolution along a trajectory in τ to generate a new four-dimensional field configuration
3. Apply MRT accept/reject test to MD discretization error

Backup: Failure of Leibnitz rule in discrete space-time

Given that $\left\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\right\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = 2i\sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu$ is problematic,

why not try $\left\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\right\} = 2i\sigma^\mu_{\alpha\dot{\alpha}} \nabla_\mu$ for a discrete translation?

Here $\nabla_\mu \phi(x) = \frac{1}{a}\left[\phi(x + a\widehat{\mu}) - \phi(x)\right] = \partial_\mu \phi(x) + \frac{a}{2}\partial_\mu^2 \phi(x) + \mathcal{O}(a^2)$

Essential difference between ∂_μ and ∇_μ on the lattice, $a > 0$

$$\begin{aligned} \nabla_\mu \left[\phi(x)\chi(x)\right] &= a^{-1}\left[\phi(x + a\widehat{\mu})\chi(x + a\widehat{\mu}) - \phi(x)\chi(x)\right] \\ &= \left[\nabla_\mu \phi(x)\right]\chi(x) + \phi(x)\nabla_\mu \chi(x) + a\left[\nabla_\mu \phi(x)\right]\nabla_\mu \chi(x) \end{aligned}$$

We only recover the Leibnitz rule $\partial_\mu(fg) = (\partial_\mu f)g + f\partial_\mu g$ when $a \to 0$

$\Longrightarrow$ “Discrete supersymmetry” breaks down on the lattice

(Dondi & Nicolai, “Lattice Supersymmetry”, 1977)

Backup: Twisting $\longleftrightarrow$ Kähler–Dirac fermions

The Kähler–Dirac representation is related to the spinor $Q_\alpha^{\mathrm{I}}, \overline{Q}_{\dot\alpha}^{\mathrm{I}}$ by

$$\begin{pmatrix} Q_\alpha^1 & Q_\alpha^2 & Q_\alpha^3 & Q_\alpha^4 \\ \overline{Q}_{\dot\alpha}^1 & \overline{Q}_{\dot\alpha}^2 & \overline{Q}_{\dot\alpha}^3 & \overline{Q}_{\dot\alpha}^4 \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu\gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu\gamma_\mu\gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a\gamma_a + \mathcal{Q}_{ab}\gamma_a\gamma_b$$

with $a, b = 1, \cdots, 5$

The 4×4 matrix involves R symmetry transformations along each row and (euclidean) Lorentz transformations along each column

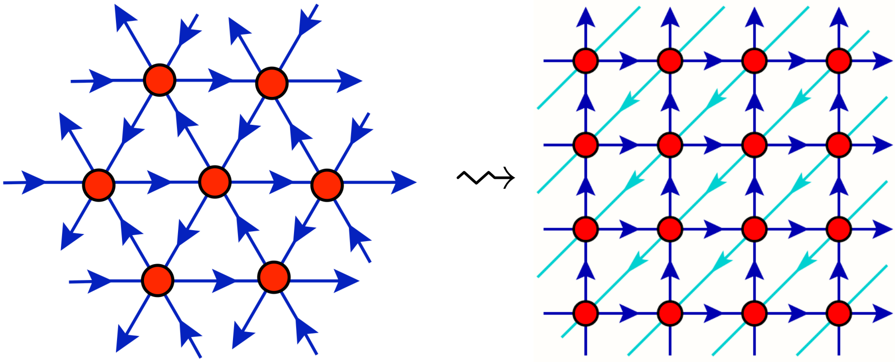
$\Longrightarrow$ Kähler–Dirac components transform under "twisted rotation group"

$$\mathrm{SO}(4)_{tw} \equiv \mathrm{diag}\left[\mathrm{SO}(4)_{\mathrm{euc}} \otimes \mathrm{SO}(4)_R\right]$$

$\uparrow$ only $\mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$

Backup: Hypercubic representation of A_4^* lattice

In the code it is very convenient to represent the A_4^* lattice
as a hypercube with a backwards diagonal



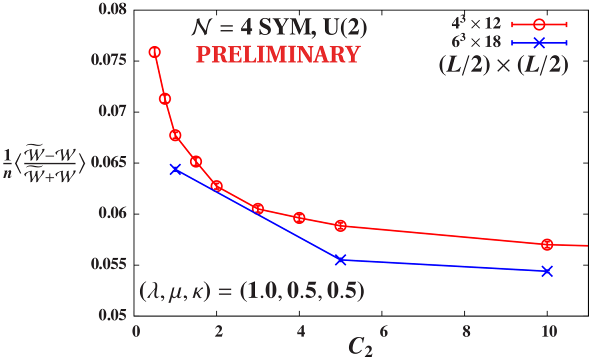
Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

Results from arXiv:1411.0166 to be revisited with the new action...

$\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ from restoration of R symmetry (motivation for A_4^* lattice)

Modified Wilson loops test R symmetries at non-zero lattice spacing

Parameter c_2 may need log. tuning in continuum limit



Backup: More on flat directions

1. Complex gauge field $\Longrightarrow$ U(N) $=$ SU(N) $\otimes$ U(1) gauge invariance
   U(1) sector decouples only in continuum limit

2. $\mathcal{Q}\,\mathcal{U}_a = \psi_a \Longrightarrow$ gauge links must be elements of algebra
   Resulting flat directions required by supersymmetric construction
   but must be lifted to ensure $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

We need to add two deformations to regulate flat directions

SU(N) scalar potential $\propto \mu^2 \sum_a \left(\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - N\right)^2$

U(1) plaquette determinant $\sim G \sum_{a<b} \left(\det \mathcal{P}_{ab} - 1\right)$

Scalar potential **softly** breaks $\mathcal{Q}$ supersymmetry

susy-violating operators vanish as $\mu^2 \to 0$

Plaquette determinant can be made $\mathcal{Q}$-invariant [arXiv:1505.03135]

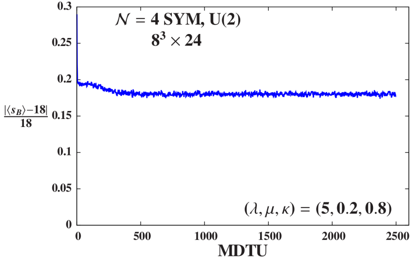
Backup: One problem with flat directions

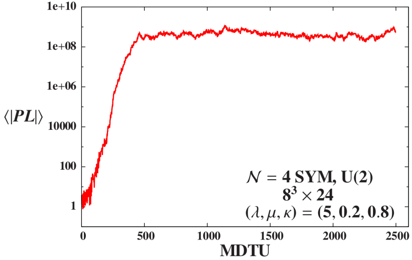
Gauge fields $\mathcal{U}_a$ can move far away from continuum form $\mathbb{I}_N + \mathcal{A}_a$ if $\mu^2/\lambda_{\text{lat}}$ becomes too small

Example for $\mu = 0.2$ and $\lambda_{\text{lat}} = 5$ on $8^3 \times 24$ volume

Left: Bosonic action is stable $\sim$18% off its supersymmetric value

Right: Polyakov loop wanders off to $\sim 10^9$

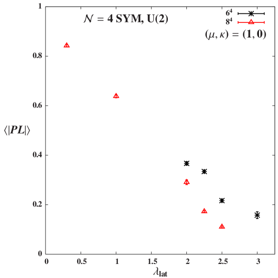


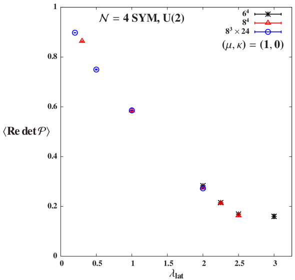


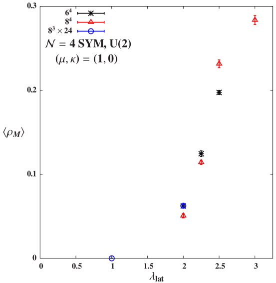
Backup: Another problem with U(1) flat directions

Can induce monopole condensation $\longrightarrow$ transition to confined phase

This lattice phase is not present in continuum $\mathcal{N} = 4$ SYM







Around the same $\lambda_{\text{lat}} \approx 2\ldots$

Left: Polyakov loop falls towards zero

Center: Plaquette determinant falls towards zero

Right: Density of U(1) monopole world lines becomes non-zero

Backup: More on soft susy breaking

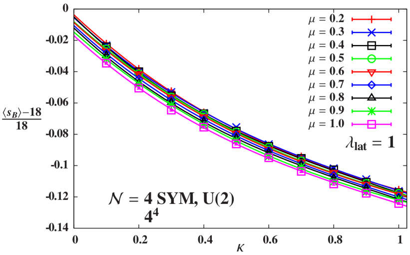
Before 2015 we used a more naive constraint on plaquette det.:

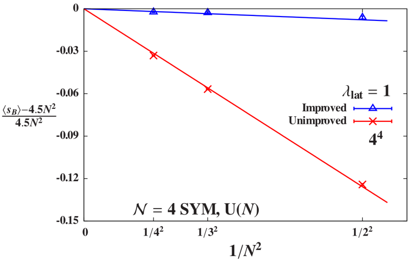
$$S_{soft} = \frac{N}{2\lambda_{\text{lat}}}\mu^2 \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2 + \kappa \sum_{a<b} \left|\det \mathcal{P}_{ab} - 1\right|^2$$

Both terms softly break $\mathcal{Q}$ but $\det \mathcal{P}_{ab}$ effects dominate

Left: The bosonic action provides another Ward identity $\langle s_B \rangle = 9N^2/2$

Right: Soft susy breaking is also suppressed $\propto 1/N^2$





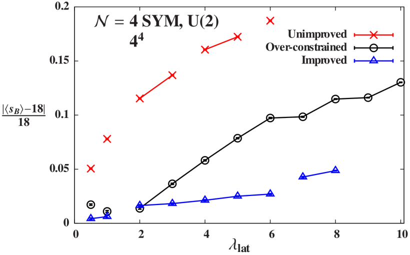
Backup: More on supersymmetric constraints

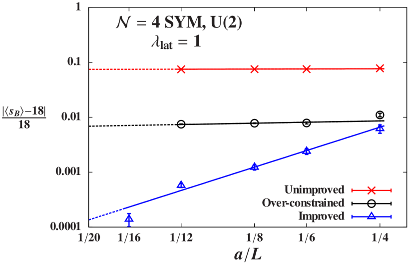
arXiv:1505.03135 introduces method to impose $\mathcal{Q}$-invariant constraints

Basic idea: Modify aux. field equations of motion $\longrightarrow$ moduli space

$$d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) \qquad \longrightarrow \qquad d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

Putting both plaquette determinant and scalar potential in $\mathcal{O}(n)$
over-constrains system $\longrightarrow$ sub-optimal Ward identity violations



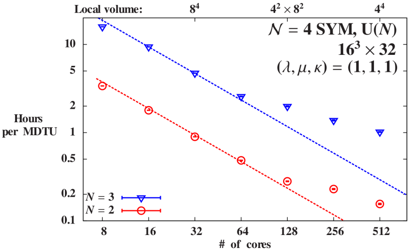


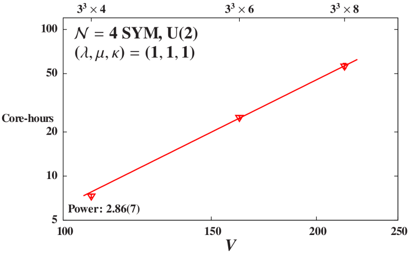
Backup: Code performance—weak and strong scaling

Results from arXiv:1410.6971 for the pre-2015 ("unimproved") action

Left: Strong scaling for U(2) and U(3) $16^3 \times 32$ RHMC

Right: Weak scaling for $\mathcal{O}(n^3)$ pfaffian calculation (fixed local volume)
$n \equiv 16N^2L^3N_T$ is number of fermion degrees of freedom





Dashed lines are optimal scaling

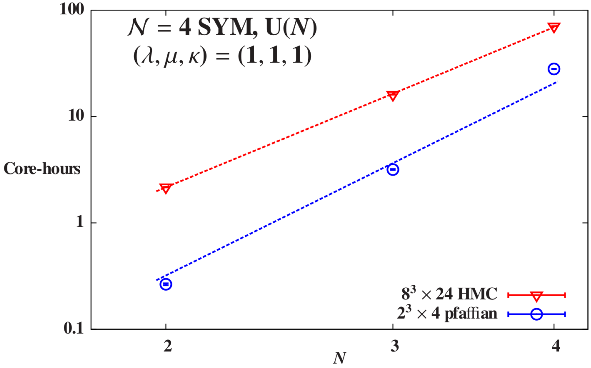
Solid line is power-law fit

Backup: Numerical costs for $N = 2$, 3 and 4 colors

Red: Find RHMC cost scaling $\sim N^5$ — recall adjoint fermion d.o.f. $\propto N^2$

Blue: Pfaffian cost scaling consistent with expected N^6

Additional factor of $\sim 2\times$ from new improved action

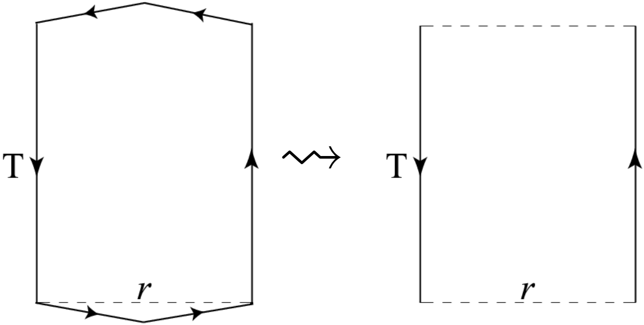


Backup: $\mathcal{N} = 4$ SYM static potential from Wilson loops

Extract static potential $V(r)$ from $r \times T$ Wilson loops

$$W(r,T) \propto e^{-V(r)\,T} \qquad\qquad V(r) = A - C/r + \sigma r$$

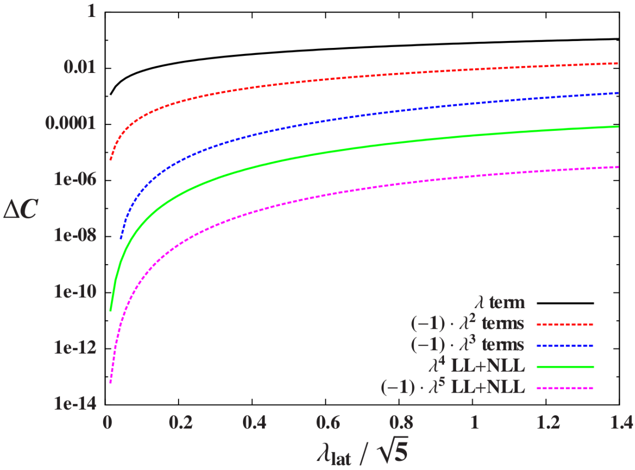
Coulomb gauge trick from lattice QCD reduces A_4^* lattice complications



Backup: Perturbation theory for Coulomb coefficient

For range of couplings currently being studied

(continuum) perturbation theory for $C(\lambda)$ is well behaved

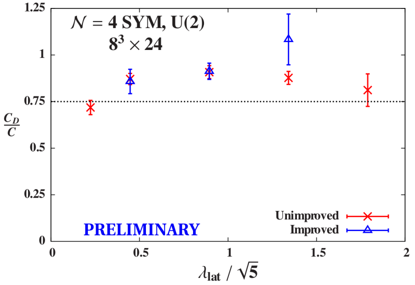


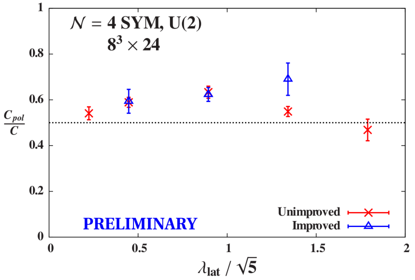
Backup: More tests of the U(2) static potential

Left: Projecting Wilson loops from U(2) $\longrightarrow$ SU(2)

$\Longrightarrow$ factor of $\frac{N^2-1}{N^2} = 3/4$

Right: Unitarizing links removes scalars $\Longrightarrow$ factor of $1/2$





Some results slightly above expected factors,

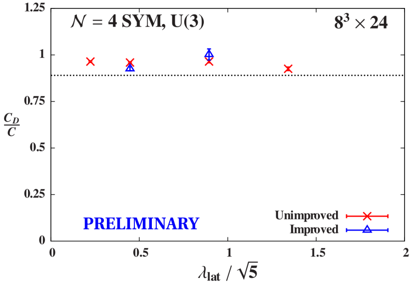
may be related to non-zero auxiliary couplings μ and κ / G

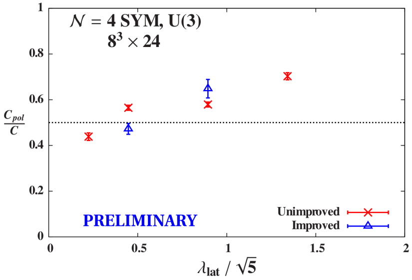
Backup: More tests of the U(3) static potential

Left: Projecting Wilson loops from U(3) $\longrightarrow$ SU(3)

$\Longrightarrow$ factor of $\frac{N^2-1}{N^2} = 8/9$

Right: Unitarizing links removes scalars $\Longrightarrow$ factor of $1/2$





Some results slightly above expected factors,

may be related to non-zero auxiliary couplings μ and κ / G

Backup: Smearing for Konishi analyses

As in glueball analyses, use smearing to enlarge operator basis

Using APE-like smearing: $(1-\alpha)$— $+$ $\frac{\alpha}{8}\sum\sqcap$,

with staples built from unitary parts of links but no final unitarization

(unitarized smearing — e.g. stout — doesn't affect Konishi)

Average plaquette is stable upon smearing (**right**)

while minimum plaquette steadily increases (**left**)